THE APRIL MEETING OF THE SOCIETY AT CHICAGO.

THE thirty-seventh regular meeting of the Chicago Section, being the sixth regular meeting of the Society at Chicago, was held at the University of Chicago, on Friday and Saturday, April 21 and 22. The meeting was attended by about sixty-five persons, among whom were the following forty-six members of the Society:

Professor R. M. Barton, Dr. Josephine E. Burns, Professor D. F. Campbell, Professor R. D. Carmichael, Dr. E. W. Chittenden, Dr. G. R. Clements, Professor H. E. Cobb, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. W. W. Denton, Professor L. E. Dickson, Professor A. Dresden, Professor W. B. Ford, Professor Tomlinson Fort, Professor A. B. Frizell, Professor A. S. Hathaway, Professor E. R. Hedrick, Dr. Cora B. Hennel, Professor W. C. Krathwohl, Dr. W. V. Lovitt, Professor A. C. Lunn, Professor W. D. MacMillan, Dr. T. E. Mason, Professor G. A. Miller, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. A. L. Nelson, Professor C. I. Palmer, Dr. A. Pell, Professor H. L. Rietz, Professor W. H. Roever, Professor D. A. Rothrock, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Professor R. B. Stone, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. VanVleck, Professor E. J. Wilczynski, Professor K. P. Williams, Dr. C. H. Yeaton, Professor J. W. A. Young.

The sessions were presided over by Professor W. B. Ford, chairman of the Section, relieved during the session of Saturday forenoon by Professor E. R. Hedrick, vice-president of the Society. Forty-four persons were present at the dinner on Friday evening.

The following papers were presented at this meeting:

(1) Professor D. M. SMITH: "Jacobi's condition for the problem of Lagrange in the calculus of variations."

(2) Professor C. H. SISAM: "On a configuration on certain surfaces."

(3) Professor T. R. RUNNING: "A new method for deriving weir formulas."

(4) Professor ARNOLD EMCH: "A new configuration on an elliptic cubic and its group of order 16."

(5) Professor L. L. DINES: "A characteristic property of self-projective curves."

(6) Dr. A. L. NELSON: "Plane nets with equal invariants."

(7) Dr. A. L. NELSON: "Quasi-periodic asymptotic plane nets."

(8) Dr. E. W. CHITTENDEN: "A theorem in general analysis."

(9) Dr. E. W. CHITTENDEN: "On the equivalence of écart and voisinage."

(10) Professor R. D. CARMICHAEL: "On the asymptotic character of functions defined by series of the form $\Sigma c_n g(x+n)$."

(11) Professor W. D. MACMILLAN: "A theorem relating to irrational numbers."

(12) Professor W. D. MACMILLAN: "A reduction of certain differential equations of the second order to algebraic types."

(13) Professor TOMLINSON FORT: "A class of developments in orthogonal functions."

(14) Dr. R. L. BORGER: "On the Cauchy-Goursat theorem."

(15) Professor H. T. BURGESS: "A practical method for determining elementary divisors."

(16) Dr. J. O. HASSLER: "Plane nets periodic of period 3 under the Laplacian transformation."

(17) Professor A. S. HATHAWAY: "The expansion, in terms of the coefficients of an equation, of the homogeneous products of the roots as a whole, and when restricted to k roots in each term."

(18) Professor W. B. FORD: "A theorem in the calculus of residues."

(19) Professor G. A. MILLER: "Graphical method of finding the possible sets of independent generators of an abelian group."

(20) Professor A. B. FRIZELL: "Postulates of continuity for arithmetic."

(21) Professor K. P. WILLIAMS: "Concerning Hill's derivation of the Lagrange equations of motion."

(22) Professor HENRY BLUMBERG: "On convex functions."

(23) Professor HENRY BLUMBERG: "On a theorem of Kempner concerning transcendental numbers."

(24) Professor E. B. VAN VLECK: "On the composition of non-loxodromic substitutions."

(25) Mr. TOBIAS DANTZIG: "A geometrical treatment of plane transformations."

(26) Professor E. H. MOORE: "On a definition of the concept: limit of a function."

(27) Mr. A. ELMENDORF: "A differentiating machine."

(28) Mr. A. M. HARDING: "On certain loci projectively connected with a given plane curve."

(29) Miss PAULINE SPERRY: "Properties of a certain projectively defined two-parameter family of curves on a general surface."

(30) Dr. C. H. YEATON: "Surfaces characterized by certain special properties of their directrix congruences."

Dr. Hassler, Mr. Harding and Miss Sperry were introduced by Professor Wilczynski, Mr. Dantzig by Professor Davisson, and Mr. Elmendorf by Professor Dresden. The papers of Professors Smith, Running, Dines, Burgess and Blumberg, and Dr. Borger were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. For the problem of Lagrange, in the calculus of variations, the Euler rule, Weierstrass condition, and corner point condition are obtainable without the use of the second variation. The Jacobi condition, however, is derived by the use of complicated transformations of the second variation or with the exclusion of important special cases. In the paper of Professor D. M. Smith it is shown that the well-known necessary condition for a minimum—i. e., that the second variation must be positive or zero for every set of admissible variations—implies a problem of Lagrange of precisely the same type as the original problem. An application of the Euler rule, Weierstrass condition, and corner point condition to this new problem leads to simple and inclusive proofs of the Legendre and Jacobi conditions for the original problem. The paper will appear in the *Transactions*.

2. Professor Sisam's paper appeared in full in the May BULLETIN.

3. The following method for deriving the formulas for the rate of flow of water over weirs avoids the necessity of a large number of experiments and can be applied to weirs of any head.

A tank having a constant horizontal cross-section has a weir fitted with a gate which can easily be removed to allow the flow of water. A chronograph connected electrically with a float and also with a vibrating pendulum records time to hundredths of a second and head to thousandths of a foot. Professor Running experimented with a right-angled notch weir, head from 2 ft. to 1.7 ft., and obtained for the law connecting time and head

$$T = -23 + 62.2H^{-1.47}.$$

T and H must satisfy the relation

$$QdT = -SdH,$$

where Q represents the rate of outflow and S the horizontal cross-section of the tank. From these two equations the formula for the weir was found to be

$$Q = 2.52H^{2.47}.$$

This represents quite closely the results of experiments performed by other methods.

The above is part of a paper to be published jointly with Professor King.

4. It is well known that a plane cubic may be generated in an infinite number of ways by projective quadratic pencils of straights with a self-corresponding element. The vertices of these pencils are points of a Steinerian couple. A Steinerian quadruple on an elliptic cubic is formed by the points of tangency of the four tangents from a point of the elliptic cubic to the same cubic. The quadruple contains six couples, and, consequently, six projectivities of quadratic pencils may be formed which may be written in 12 different ways, and which all generate the same cubic. Applying a certain cyclic process to the equations of these projectivities and making use of a certain theorem in algebra, Professor Emch obtains a configuration of 48 straights, which in triplets are concurrent

in 16 points. From one triplet all the others are obtained by the substitutions of a group of order 16 with 15 subgroups of order 2. The paper will be published in the *Annals of Mathematics*.

5. Self-projective curves, or W-curves, were defined by Klein and Lie to be those curves which admit infinitesimal projective transformations into themselves. In this paper, Professor Dines determines for these curves a simple characteristic property somewhat similar to the Steinerian definition of conics. The paper will appear in the *Annals of Mathematics*.

6. Wilczynski has treated the projective differential geometry of plane nets, as well as that of curved surfaces. Dr. Nelson sets up, in this paper, a perspective connection between these two theories by means of the following theorem of Koenigs: The asymptotic curves of a surface are projected from a fixed point on a fixed plane in a net with equal Laplace-Darboux invariants, and conversely, a plane net with equal invariants may be considered as the perspectives, from a fixed point, of the asymptotic curves of a surface. The paper will be offered for publication to the *Palermo Rendiconti*.

7. Dr. Nelson's second paper appears in the present number of the BULLETIN.

8. Dr. Chittenden completes the proof of the theorem: Every class $\mathfrak{M}$ with the composite property $D_1K_{12}\Delta$ has the property K_{12*}, and $\mathfrak{M}_*$ has the composite property

$$LCD_1A\Delta K_{12}K_{12*}.$$

The proof is accomplished with the aid of a theorem of Professor A. D. Pitcher which differs from the present theorem in that the hypothesis contains the additional property B_3. B_3 is a special property defined in terms of Δ and boundedness from zero.

Professor Pitcher in a discussion of the complete existential theory of the eight properties L, C, D, A, K_1, K_2, Δ, K_{12*}, left unsettled the question of the existence of three composite properties; viz.,

$$L^-C^-DA\Delta K_1K_2^-K_{12*}; \qquad LC^-DA\Delta K_1K_2^-K_{12*};$$
$$LC^-D^-A\Delta K_1K_2^-K_{12*}.$$

The theorem of this paper shows that the first two are non-existent, and that the third is non-existent for any system $(\mathfrak{P}; \Delta; \mathfrak{M}^{D_1})$. This extends the results obtained by Professor Pitcher who showed that the three properties were non-existent for any system $(\mathfrak{P}; \Delta; \mathfrak{M}^{D_1 B_3})$. It is understood that the development Δ is finite.

9. Hahn has shown that if a voisinage $V(a, b)$ is defined for a class $\mathfrak{Q}$ containing at least two elements, then for every element q of $\mathfrak{Q}$ there exists a continuous function μ^q, vanishing only at q and assuming values between zero and one. Dr. Chittenden modifies the existence proof of Hahn and secures a type of uniformity among the functions μ^q. Defining (a, b) as the least upper bound of $|\mu^q(a) - \mu^q(b)|$ for all elements q of $\mathfrak{Q}$, it appears that (a, b) is an écart and is defined for every pair of elements of $\mathfrak{Q}$. By means of the uniformity mentioned it is shown that $L_n(a_n, a) = 0$ implies $L_n V(a_n, a) = 0$ and that $L_n V(a_n, a) = 0$ implies $L_n(a_n, a) = 0$. Hence voisinage is equivalent to écart. This result was anticipated by Fréchet on the ground that no theorem was known for an écart which had not been established for a voisinage. This paper will be offered to the *Transactions* for publication.

10. In a previous paper (read before the Society at the recent meeting in Columbus) Professor Carmichael laid the foundations of a general theory of the series

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(x + n),$$

where $g(x)$ is a function having, in a sector V, the Poincaré asymptotic representation

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right),$$

$P(x)$ and $Q(x)$ being polynomials in x. In the present paper he shows that a function $\Omega(x)$ defined by the foregoing series satisfies the relations

$$\lim \{g(x + s)\}^{-1}\{\Omega(x) - \sum_{n=0}^{s} c_n g(x + s)\} = 0 \qquad (s = 0, 1, 2, \cdots),$$

provided that x approaches infinity in an appropriately deter-

mined sector $\bar{V}$ lying in V. Guided by this result he introduces a generalization of Poincaré's notion of asymptotic representation and discusses the character of this generalized representation, especially in relation to that of Poincaré.

11. In celestial mechanics, series occur very frequently which contain denominators having factors of the form $(i - j\gamma)$ where γ is an irrational number; i and j are integers which take all possible values. The study of series in which this phenomenon occurs has led Professor MacMillan to the following theorem:

If γ is a positive number, and if p_n/n is a rational fraction such that $| p_n - n\gamma | \leqq \frac{1}{2}$, and if

$$G_n = \left(\prod_{k=1}^{n} | p_k - k\gamma | \right)^{1/n}$$

is the geometric mean of the first n of the quantities $| p_k - k\gamma |$, then the limit of G_n, as n increases without limit, is zero if γ is rational, and is equal to $1/2e$, where $e = 2.71828 \cdots$ is the Naperian base, if γ is an irrational number which satisfies the condition that

$$a_{n+1} \leqq M q_n (q_n + 1) \cdots (q_n + s),$$

where γ expressed as a continued fraction is

$$\gamma = a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} ,$$

where q_n is the denominator of the nth principal convergent, and s is any assigned positive integer independent of n. If γ is an irrational number which does not satisfy this condition then G_n for large values of n oscillates between zero and $1/2e$.

This paper will appear in the *American Journal of Mathematics*.

12. In two former papers Professor MacMillan has shown that certain differential equations can be reduced by a linear-transcendental substitution to algebraic forms. The first of these papers dealt with a "general case," the second with equations of the first order.

The exceptions to the general case include most of the equations of dynamics, and therefore these equations require

further study. The present investigation takes up differential equations of the second order

$$\frac{dx_i}{dt} = \alpha_i x_i + \text{terms of higher degree} \quad (i = 1, 2),$$

where all of the coefficients of the right members are constants, and where the α_i considered as points in the complex plane lie on a straight line through the origin and on opposite sides of the origin. If these conditions on the α_i are not satisfied the equations belong to the general case.

If the ratio α_2/α_1 is an irrational number of a certain type (which includes all irrational algebraic numbers) then the differential equations can be reduced by a linear-transcendental substitution to its linear terms, and the substitution is convergent if the differential equations are convergent.

If the ratio α_2/α_1 is rational there is no essential restriction in supposing that $\alpha_1 = +1$, $\alpha_2 = -1$. In this case the differential equations cannot, in general, be reduced to their linear terms. They can however be reduced to a set of differential equations which are algebraic, and which are easily integrable. The convergence of the substitution is assured if the equations are canonical

$$x_1' = \frac{\partial H}{\partial x_2}, \qquad x_2' = -\frac{\partial H}{\partial x_1},$$

where H is a convergent power series in x_1 and x_2. In certain other cases, however, the substitution may diverge even though the differential equations converge. The paper will be offered to the *Transactions*.

13. Professor Fort's paper considers the development of an arbitrary function $f(x)$ in infinite series of solutions of

$$(1) \qquad \frac{d}{dx}[k(x)y'(x)] - [\lambda^2 g(x) + l(x)]\, y = 0,$$

satisfying the conditions (2) $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$; and secondly in terms of solutions satisfying the conditions (3) $y(0) = -y(2\pi)$, $y'(0) = -y'(2\pi)$. The principal results are given in the following theorem.

Let $(k)x$ and $g(x)$ be positive at all points and have second derivatives integrable from 0 to 2π. Let $l(x)$ have a first

derivative integrable from 0 to 2π and let $f(x)$ itself be integrable from 0 to 2π. Let moreover $g(0)k(0) = g(2\pi)k(2\pi)$ and $[g(0)k(0)]' = [g(2\pi)k(2\pi)]'$. Then the development corresponding to $f(x)$ in terms of solutions of (1) satisfying (2) (more accurately described in the paper) converges at any particular point of (4) $0 \leqq x \leqq 2\pi$, when and only when the Fourier series for $f(x)$ converges at that point and to the same value. It converges uniformly over the whole interval or any subinterval when and only when this is true of the Fourier series and diverges to ∞ or $-\infty$ at any particular point when and only when this is true of the Fourier series. Moreover it is summable by the method of the arithmetic mean when and only when the same thing is true of the Fourier series and to the same value.

Secondly: The development in terms of the solutions of (1) satisfying (3) bears the same relation to the development of $f(x)$ in terms of $\sin [(2n+1)/2]x$ and $\cos [(2n+1)/2]x$, $n = 0, 1, 2, \cdots$ as is borne by the series just discussed to the Fourier series.

14. In this paper Dr. Borger shows that if the two functions $U(x, y)$, $V(x, y)$ satisfy the following conditions in a region R:

(1) U and V are continuous in (x, y),

(2) U_x', V_x', U_y', V_y' exist and are finite,

(3) $U_x' = V_y'$; $U_y' = -V_x'$,

(4) U and V possess proper total differentials;

then U and V are analytic functions of the real variables x, y.

An immediate corollary is:

If $w = f(z) = U(x, y) + iV(x, y)$ possess a finite derivative at every point of a region R:

(a) This derivative is a continuous function of z.

(b) All the derivatives of w exist.

(c) w can be represented by a power series in z.

This paper will be offered to the BULLETIN for publication.

15. In this paper Professor Burgess shows that there exists a very simple relation between the exponents of the elementary divisors of the characteristic matrix $\lambda I + B$ which are connected with the linear factor $\lambda - \alpha$ of the characteristic equation $|\lambda I + B| = 0$, and the ranks of the matrices $(\alpha I + B)^n$ for $n = 1, 2, 3$, etc. In fact these ranks determine the elementary divisors and conversely.

It is also shown that the method is easily applied to any matrix of the form $\lambda A + B$ whose determinant does not vanish identically in λ. The paper will be published in the *Annals of Mathematics*.

16. In the projective theory of plane nets, as developed by Wilczynski, the members $y_1(u, v)$, $y_2(u, v)$, $y_3(u, v)$ of any fundamental system of solutions of a completely integrable system of partial differential equations of the form

$$(1) \qquad \begin{aligned} y_{uu} &= ay_u + by_v + cy, \\ y_{uv} &= a'y_u + b'y_v + c'y, \\ y_{vv} &= a''y_u + b''y_v + c''y, \end{aligned}$$

are interpreted as the homogeneous coordinates of a point P_y in a plane, defining a non-degenerate net of plane curves. The invariants and covariants of the system (1) under the transformations $y = \lambda(u, v)\bar{y}$ and $\bar{u} = U(u)$, $\bar{v} = V(v)$ may be interpreted geometrically by certain projective properties of the net. The two covariants $\rho = y_u - b'y$, $\sigma = y_v - a'y$ give rise to the homogeneous coordinates of two points P_ρ and P_σ when we substitute successively for y the values y_1, y_2, y_3. As u and v vary P_σ and P_ρ describe nets called the first and minus first Laplacian transforms of the original net. The minus first Laplacian transform of the σ-net and the first Laplacian transform of the ρ-net both coincide with the original y-net.

In this paper Dr. Hassler considers classes of nets such that the first Laplacian transform of the σ-net is the ρ-net and the minus first Laplacian transform of the ρ-net is the σ-net. Such nets are periodic of period 3 under the Laplacian transformation. He establishes necessary and sufficient conditions for such periodicity, determines the form of the coefficients of system (1) for such a net, computes its invariants and covariants, and studies the osculating conics of the curves of the original net and of its Laplacian transforms. The determination of a net of this kind by certain boundary conditions is considered and the following theorem is proved:

Choose an arbitrary triangle LMN. Through the point L pass two non-rectilinear but otherwise arbitrary analytic curves C and C', tangent to LN and LM, respectively, at L. Through each of the points M and N pass another such arbi-

trary curve, C_1' through M and C_{-1} through N, tangent to MN at M and N, respectively. Select any conic M_1 tangent to LM at M and a second conic N_{-1} tangent to LN at N. There exists one and only one net which is periodic of period 3 under the Laplacian transformation, which contains the given curves C and C', which has C_{-1} corresponding to C and C_1' corresponding to C' as curves of the minus first and first Laplacian transforms respectively, and which, moreover, has M_1 as the osculating conic of the curve of the first Laplacian transform corresponding to C and N_{-1} as the osculating conic of the curve of the minus first Laplacian transform corresponding to C'.

This paper will appear in the *Palermo Rendiconti.*

17. In this paper, Professor Hathaway shows, from the properties of gamma coefficients defined by

$$(A\alpha B\beta) = (A\alpha + B\beta)\Gamma(\alpha + \beta)/\Gamma(\alpha + 1)\Gamma(\beta + 1), \text{ etc.},$$

that the function of weight n in the coefficients of the equation

$$x^m = a_1x^{m-1} + \cdots + a_m$$

defined by

$$F(n) = \Sigma(A_1\alpha_1A_2\alpha_2 \cdots A_n\alpha_n)a_1^{\alpha_1}a_2^{\alpha_2} \cdots a_n^{\alpha_n}$$

(the summation being for all integral solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n)$$

is the unique solution of the difference equation

$$F(n) = a_1F(n-1) + \cdots + a_{n-1}F(1) + A_na_n.$$

When $n' > m$, $a_{n'} = 0$, in the above values of $F(n)$.

Hence, the problem of expanding any symmetric function of the roots is the problem of determining the coefficient A_n in its difference equation of the above type. Thus, for $\pi(n)$, the homogeneous products, $A_n = 1$, i. e., the numerical coefficient of any term in $\pi(n)$ is the multinomial coefficient of its exponents $(1\alpha_1 1\alpha_2 \cdots 1\alpha_n)$.

Also, for the homogeneous products, one root in a term, $\pi_1(n) = s_n$, by Newton's formulas, $A_n = n$. The resultant coefficient of any term $(1\alpha_1 2\alpha_2 \cdots n\alpha_n)$ is identical with Waring's.

To determine A_n for the homogeneous products, k roots

at a time, $\pi_k(n)$, we have only to consider the equation $x^n = 1$, and find $A_n = (-1)^{k+1}[1(n-k)1k]$, a binomial coefficient.

There is a remarkable relation between the homogeneous products of all weights, and those of weight n and all orders k. Thus, expanding $F(n)$ in terms of the arbitrary quantities A, we obtain

$$F(n) = A_1a_1\pi_{n-1} + A_2a_2\pi_{n-2} + \cdots + A_na_n.$$

Thus we have the table,

	$a_1\pi_{n-1}$	$a_2\pi_{n-2}$	$a_3\pi_{n-3}$	$a_4\pi_{n-4}$	$\cdots$
$\pi(n) =$	1	1	1	1	$\cdots$
$\pi_1(n) =$	1	2	3	4	$\cdots$
$-\pi_2(n) =$		1	3	6	$\cdots$
$\pi_3(n) =$			1	4	$\cdots$
$-\pi_4(n) =$				1	$\cdots$
$\cdots$					

The relation may also be written as an identity in an arbitrary variable x,

$$\Sigma(-x)^k\pi_k(n) + \Sigma(1+x)^k a_k\pi(n-k) = \pi(n) \qquad (k = 1, 2, \cdots, n),$$

and by substituting $-1-x$ for x we see that the relation is symmetrical.

18. Cauchy's integral theorem for functions of a complex variable has often been employed to advantage for the summing of infinite series, such applications belonging to the subject commonly known as "the calculus of residues." After pointing out the limitations naturally present in using this method of summation, Professor Ford's paper proceeds to outline a way by which the fundamental difficulty may be avoided in many cases, and an actual formula for sum is obtained which has a wide range of applicability. The paper will be offered for publication to the BULLETIN.

19. The term independent generators of an abelian group has two distinct meanings. According to one of these meanings the set of operators $s_1, s_2, \cdots, s_\lambda$ is said to be a set of independent generators of the group G provided these operators generate G but no $\lambda - 1$ of them generate G. According to the other meaning such a set is called a set of independent generators of G if and only if the group generated by every $\lambda - 1$ of them has only identity in common with the group generated by the remaining one. The value of λ is the same in both cases. When the term independent generator is used with the former meaning it is said to be used in the general sense and when it is used with the latter meaning it is said to be used in the restricted sense. In the general sense the operators which cannot be used as independent generators constitute a subgroup known as the ϕ-subgroup. In the restricted sense these operators do not always constitute a subgroup.

Professor Miller's results relate to the use of independent generators in the restricted sense. If the order of an abelian group G is p^m, p being a prime number, and if we form the quotient group of G with respect to its subgroup H composed of the pth power of each of its operators, the order of this quotient group being p^λ, then λ is the number of the independent generators of G. The number of operators of G which can be used as independent generators of lowest order is equal to the number of operators of lowest order in the cosets of G with respect to H. The totality of the operators which can be used as independent generators of other orders may be determined similarly.

20. The continuity axioms of Cantor and Weierstrass are essentially assumptions about the existence of limits. Professor Frizell submits a set of postulates based on ideas of order, from which it is easy to deduce both Cantor's assumption and the principle of Archimedes, and therefore also the postulates of Weierstrass and Dedekind. This paper will be offered to the BULLETIN for publication.

21. Professor Williams's paper appears in the present number of the BULLETIN.

22. A real function $f(x)$ uniquely defined in the interval (a, b)

is said to be "convex," if for every two values x_1 and x_2 in the interval (a, b) the inequality

$$f\left(\frac{x_1 + x_2}{2}\right) \leqq \frac{f(x_1) + f(x_2)}{2}$$

holds.* From a paper of Bernstein and Doetsch,† according to which a convex function is either continuous (except possibly at the ends of the interval) or totally discontinuous, it appears that discontinuous convex functions are of relatively minor importance. Let then the continuity of $f(x)$ be assumed henceforth. Jensen has shown‡ that the derivative of $f(x)$ may be non-existent in an everywhere dense set of points, but the right-hand derivative (together with the left-hand derivative) always exists. Professor Blumberg proves (among other things): (*a*) The right-hand derivative of $f(x)$ is a monotone increasing function; conversely, if the right-hand derivative of a continuous function $g(x)$ exists everywhere and is a monotone increasing function, then $g(x)$ is convex. (*b*) The right-hand derivative of $f(x)$ is continuous on the right at every point of (a, b). (*c*) The points of (a, b) at which the derivative of $f(x)$ is non-existent form at most a denumerable set (conversely, given a denumerable set S, a convex function $f(x)$ may be constructed, such that the derivative of $f(x)$ is existent at every point not in S and non-existent at every point in S). (*d*) Every continuous convex function is the indefinite integral of a monotone increasing function; conversely, the indefinite integral of a monotone increasing function is convex (and continuous).

The proofs are extremely simple. Moreover, the proofs of most of the theorems (on continuous convex functions) contained in the two papers quoted above are simplified and rendered nearly intuitionally evident; the Cauchy artifice,§ which is at the basis of the proofs of Jensen, Bernstein and Doetsch, may be altogether dispensed with.

23. The theorem in question is as follows:‖ The number

$$f(x) = \Sigma_{r=0}^{r=\infty} \alpha_r x^r / a^{c^r}$$

* See Jensen, *Acta Math.*, vol. 30, p. 176.
† *Math. Ann.*, 1915, p. 514.
‡ L. c.
§ See Jensen, l. c., p. 175.
‖ See BULLETIN, Mar., 1915, p. 285.

is transcendental for every rational (real) x ($\neq 0$), if a and c are integers $\geqq 2$ and if (1) $|\alpha_r| < M^r$, M arbitrary but fixed, and (2) only a finite number of the α's vanish. Professor Blumberg shows by means of a modification of Dr. Kempner's method that the condition (2) may be substantially altogether dropped and replaced by the restriction (evidently to be demanded) that $f(x)$ shall not break off after a finite number of terms. Condition (1) may be replaced by the milder condition $|\alpha_r| < M^{r^k}$, k being like M arbitrary but fixed.

25. In this paper Mr. Dantzig considers a continuous plane n to 1 transformation T. Let C be an arbitrary curve through a point P and having t for tangent at P and let C_1P_1 be the images of C and P, and t_1 the tangent to C_1 at P_1; t and t_1 will meet in a point τ. The locus of the point τ for the totality of the curves C through P is a conic, which the author calls the indicatrix of T for the point P. There exists for each point in the plane such an indicatrix I, and the totality of the indicatrices I form a two-parameter system of conics. It is further shown that the behavior of T at a point P is characterized by the indicatrix at P. In particular if the totality of the indicatrices form a system of circles or equilateral hyperbolas, the transformation is conformal.

The case of the degeneracy of the indicatrix is further taken up and the intimate relation of the indicatrix to the invariant elements of T is brought out. It is shown that in the case of the identical degeneracy of the indicatrix either T is a pseudo transformation, or there exists an infinity of invariant straight lines enveloping an absolutely invariant curve.

The value of the method for the classification of plane transformations is discussed. Applied to collineations the method furnishes a purely geometrical basis for projective geometry independent of any consideration of anharmonic ratios, elliptic involutions or polar reciprocity.

The author applies this method to a great number of problems in geometry, particularly to the proof of Poincaré's last theorem, to the problem of closure and the theory of unicursal plane curves.

26. By means of a remarkably simple generalization of the concept limit of a function Professor E. H. Moore has recently found it possible to define the integration process J in a new

theory of linear integral equations in general analysis, a general theory having as its guiding instance a body of theorems, due to Hilbert and developed more directly by Hellinger, concerning limited linear, quadratic and hermitian forms in infinitely many numerical variables. In the present paper, which will be offered to the *Transactions*, Professor Moore gives definitions of the new limit and of various allied concepts, proves a number of fundamental propositions involving these concepts, and points out the sense in which various classical limits are instances of the new limit. The definition is as follows: Consider a class $\mathfrak{Q}$ of elements q and a binary relation κ on $\mathfrak{Q}$ which is reflexive, transitive and of such a nature that for every two elements q_1, q_2 there is an element q_{12} of such a nature that $q_{12} \kappa q_1$ and $q_{12} \kappa q_2$. (Examples: I. $\mathfrak{Q}$ is the class of positive integers q; κ is the binary relation $\geqq$ on $\mathfrak{Q}$. II. $\mathfrak{Q}$ is the class of all finite subclasses q of a class $\mathfrak{P}$ of elements p; κ is the binary relation $\supset$, inclusion, on $\mathfrak{Q}$, viz., for two classes q_1, q_2, $q_1 \supset q_2$ denotes that every element p of q_2 is an element of q_1.) Then, with respect to this relation κ, a numerically valued function φ or $\varphi(q)$ on $\mathfrak{Q}$ has the number a as limit in case for every positive number e there exists an element q_e depending on e of such a nature that for every element $q \kappa q_e$ it is true that $| \varphi(q) - a | \leqq e$. (Accordingly, in example I, the number a is the limit in the classical sense of the numerical sequence $\varphi(1), \varphi(2), \cdots, \varphi(q), \cdots$. The limit of example II is the limit used in the definition of the integration process J; it is to be noted that this limit is of general reference in the sense that it has reference to no metric or other features of the class $\mathfrak{P}$ which is accordingly a (truly) general class.)

27. The differentiating machine designed by Mr. Elmendorf plots the differential curve of any given curve, and is primarily applicable in drawing the rate curve for any empirical curve. A silver reflector is mounted at the end of a bar and at right angles to it so that when the mirror is set upon the curve and turned until the image and the actual curve form a continuous line, the bar is tangent to the curve at the point in question. The lengths cut off by the bar on a vertical erected at one end of a horizontal link of constant length which is so arranged that the other end is over the center of the mirror, are plotted as ordinates of the desired differential curve. An

account of the machine was published in the *Scientific American Supplement* for Feb. 12, 1916.

28. A cubic which has eight consecutive points in common with a curve C at a point P is called an eight-pointic cubic of P. One of the ∞^1 eight-pointic cubics has a double point at P and is known as the eight-pointic nodal cubic. The double point tangents are, in general, distinct and form, with the inflectional line of the cubic, a triangle, which is called the canonical triangle.

The anharmonic curve which has eight consecutive points in common with C at the point P is called the osculating anharmonic curve. Associated with an anharmonic curve is, in general, a triangle, called the invariant triangle, which has the following property. Suppose any tangent is drawn to the curve. The anharmonic ratio of the point of tangency and the three points of intersection of the tangent with the sides of the invariant triangle is constant for all points on the curve.

Associated with every point P on the curve C there are six points and six lines which form the vertices and sides of the canonical triangle and the invariant triangle of the osculating anharmonic. As P moves along the curve C the six points will describe certain loci and the six lines will envelope certain curves. The object of Mr. Harding's paper is to study the character of these loci. The paper will be offered for publication to the *Giornale di Matematiche*.

29. Professor Wilczynski has shown that the theory of nonruled analytic surfaces may be based on a completely integrable system of partial differential equations of the form

$$\frac{\partial^2 y}{\partial u^2} + 2b\frac{\partial y}{\partial v} + fy = 0,$$

$$\frac{\partial^2 y}{\partial v^2} + 2a'\frac{\partial y}{\partial u} + gy = 0.$$

In the present paper, Miss Sperry associates with each point of the surface a line through the point but not in the tangent plane of the point. In this way a congruence L is determined. There are two one-parameter families of curves on the surface along which the lines of L generate developables. We have called them the congruential torsal curves. On every surface

there is a two-parameter family of curves having the property that the osculating planes of all the curves of the family passing through a point have in common the line of L through that point. Their differential equation is

$$\frac{d^2u}{dt^2}\frac{dv}{dt} - \frac{d^2v}{dt^2}\frac{du}{dt} + 2\left\{b\left(\frac{du}{dt}\right)^2 - a'\left(\frac{dv}{dt}\right)^2\right\} + 2\left\{p_1\left(\frac{du}{dt}\right)^2\frac{dv}{dt} + p_2\frac{du}{dt}\left(\frac{dv}{dt}\right)^2\right\} = 0,$$

where p_1 and p_2 depend upon the choice of L.

Conversely the integral curves of an equation of this type have the property that the osculating planes of all the curves of the family which pass through a point form a pencil. These curves we have called the congruential union curves. If L is the congruence of surface normals the torsal curves are lines of curvature and the union curves are geodesics. If L is the congruence of directrices of the second kind the torsal curves are directrix curves.

A necessary and sufficient condition that the union curves be plane is that they be torsal curves, a generalization of a well known theorem concerning geodesics and lines of curvature.

30. With every point P of a non-ruled surface S Professor Wilczynski has associated a pair of straight lines; one of these lines lies in the tangent plane and is called the directrix of the first kind, while the other pierces the surface at P and is called the directrix of the second kind. The two families of developables of the two congruences thus associated with S are determined by one and the same net of curves, the directrix curves on S.

In this paper Dr. Yeaton obtains the conditions under which the focal sheets of the directrix congruence of the second kind degenerate into curves. Limiting the discussion to surfaces whose directrix curves form a conjugate net, he shows that if this net is not degenerate, the congruence in question may be linear. The surfaces thus determined are projectively equivalent to the surface

$$z = \frac{1}{4}xy\left(\log_e\frac{x}{y}\right)^2$$

On any surface the two families of asymptotic curves are projectively equivalent, each lies on a quadric and is identically self-dual; the directrix curves are plane curves and one of the two families consists of conics. A non-degenerate quadric and a straight line, which is not a ruling of the quadric, constitute the focal surface of the directrix congruence of the first kind. The finite equations of the various associated loci are obtained.

ARNOLD DRESDEN,
Secretary of the Section.

NOTE ON FUNCTIONS OF SEVERAL COMPLEX VARIABLES.

BY PROFESSOR WILLIAM F. OSGOOD.

(Read before the American Mathematical Society, April 29, 1916.)

THE object of the present note is at once to extend the scope of a fundamental theorem of the theory of analytic functions of several complex variables and to simplify its proof.*

Definition.—Let S be the cylindrical region $(S_1, \cdots, S_n)$,

$$S_k: \qquad |z_k| < r_k \qquad (k = 1, \cdots, n);$$

let Σ be the region $(\Sigma_1, \cdots, \Sigma_n)$,

$$\Sigma_j: \qquad |z_j| < h_j < r_j \qquad (j = 1, 2);$$

$$\Sigma_k: \qquad |z_k| < r_k \qquad (k = 3, \cdots, n);$$

and let T be the region whose points are interior to S, but exterior to Σ:

$$T = S - \Sigma.\dagger$$

THEOREM. *Let $f(z_1, \cdots, z_n)$ be analytic throughout the region T. Then $f(z_1, \cdots, z_n)$ admits analytic continuation throughout S.*

* The theorem was given by Kistler, "Ueber Funktionen von mehreren komplexen Veränderlichen," § 7, Basel, 1905, for the case that the excepted points lie on a finite number of analytic manifolds, each of $n - 2$ complex dimensions, and was proven by means of n-fold integrals.

† This symbolic form is suggestive, but not quite accurate, since it would assign to T certain of its boundary points, and T consists only of interior points.

The proof is given at once by Cauchy's integral formula for functions of a single variable—for simplicity we set $n = 3$—

$$f(z_1, z_2, z_3) = \frac{1}{2\pi i}\int_C \frac{f(z_1, t_2, z_3)dt_2}{t_2 - z_2}.$$

Here C shall be a circle,

$$|t_2| = r_2', \quad h_2 < r_2' < r_2,$$

r_2' being taken as near to r_2 as one pleases. Furthermore, z_1 shall be a point of the ring

$$h_1 < |z_1| < r_1,$$

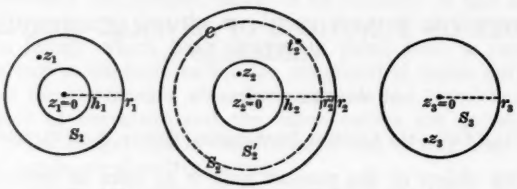

FIG. 1.

while z_3 is any point of the circle

$$|z_3| < r_3.$$

If finally z_2 is any point interior to the circle C, the hypotheses of the theorem justify the above formula.

But the integrand, for any fixed point t_2 on the circle C, is analytic throughout the whole region $S' = (S_1, S_2', S_3)$,

S_2': $$|z_2| < r_2';$$

and it is continuous when (z) lies in S' and t_2 on C. Hence the integral represents a function analytic throughout S'.

Thus $f(z_1, z_2, z_3)$ admits analytic continuation throughout S', and hence finally throughout S.

Remark. The foregoing theorem is contained in a theorem of Hartogs's.* Let a_1 be a point of the region $h_1 < |z_1| < r_1$, and let $a_3 = 0$. Then

(i) $f(z_1, z_2, z_3)$ is analytic in each point (a_1, z_2, a_3), where z_2 is any point of S_2', including the boundary;

Sitzungsber. der Münchener Akad., 36 (1906), p. 223. Cf. also the Madison Colloquium, pp. 168, 169.

(ii) $f(z_1, z_2, z_3)$ is analytic in each point (z_1, t_2, z_3), where t_2 is any point of C, and z_1, z_3 lie respectively in S_1 and S_3.

Hence $f(z_1, z_2, z_3)$ admits analytic continuation throughout S', and thus throughout S.

Hartogs's proof of the more general theorem is less simple, involving as it does n-fold integrals.

HARVARD UNIVERSITY,
April 17, 1916.

QUASI-PERIODICITY OF ASYMPTOTIC PLANE NETS.

BY DR. ALFRED L. NELSON.

(Read before the American Mathematical Society, April 21, 1916.)

1. *Introduction.*—The projective properties of plane nets of curves have been discussed by Wilczynski.* For this purpose he makes use of a certain completely integrable system of three linear homogeneous partial differential equations of the second order, namely,

$$\begin{aligned} y_{uu} &= ay_u + by_v + cy, \\ y_{uv} &= a'y_u + b'y_v + c'y, \\ y_{vv} &= a''y_u + b''y_v + c''y. \end{aligned} \tag{1}$$

Three linearly independent solutions of this system, $y^{(k)}$ $(k = 1, 2, 3)$, are interpreted as the homogeneous coordinates of a point P_y which generates the plane net. The projective properties of the net are expressed in terms of the invariants of (1) under the transformations

$$y = \lambda(u, v)\bar{y}; \qquad \bar{u} = U(u), \qquad \bar{v} = V(v). \tag{2}$$

Two of these invariants,

$$H = c' + a'b' - a_u', \quad K = c' + a'b' - b_v',$$

the so-called Laplace-Darboux invariants, are expressed entirely in terms of the middle equation, which is of the type

* Wilczynski, "One-parameter families and nets of plane curves," *Transactions Amer. Math. Society*, vol. 12 (1911), no. 4, pp. 473–510.

studied by Laplace.* A certain special case, namely, when the invariants H and K are equal, acquires especial interest in view of a theorem of Koenigs:† *The perspectives of the asymptotic curves of a surface from a fixed point on a fixed plane form a net with equal invariants. Conversely, a plane net with equal invariants may be regarded as the perspectives from a fixed point of the asymptotic curves of a surface.* We shall speak of a plane net for which $H = K$ as an *asymptotic* plane net. The present paper has as its object the discussion of the Laplace transformation, to which a large part of Wilczynski's paper is devoted, for the special case mentioned. The characteristic system of partial differential equations for the general Laplace transformed net will be computed, together with its fundamental invariants, and certain theorems concerning quasi-periodic plane nets will be deduced.

The first Laplace transform is the net generated by the covariant point $P_{y'}$, whose coordinates are $y'^{(k)} = y_v^{(k)} - a'y^{(k)}$ $(k = 1, 2, 3)$. The minus first Laplace transform is described by the covariant point $P_{y^{(-1)}}$, where $y^{(-1)} = y_u - b'y$. Each of these new nets has a first and a minus first Laplace transform, and it is readily shown that the minus first transform of the first transform, as well as the first transform of the minus first transform, is the original net. Accordingly, we have, in general, an infinite chain of nets, the Laplace suite, covariantly connected with the original net. Each net of the suite has, of course, a characteristic system of equations, and we shall call the system of the ith transform $(Y^{(i)})$ (i a positive or negative integer), and distinguish the coefficients and invariants of this transform by the subscript i.

Under the assumption of the equality of H and K, the system (1) may be put in the unique form

$$
\begin{aligned}
y_{uu} &= -2\frac{\delta_u}{\delta}\cdot y_u + \mathfrak{B}y_v - \mathfrak{B}\frac{\partial}{\partial v}\log\frac{\mathfrak{B}}{\delta^2}\cdot y,\\
(3)\qquad (Y)\quad y_{uv} &= Hy, \quad \left(H = \mathfrak{A}''\mathfrak{B} - \frac{\partial^2 \log \delta^2}{\partial u \partial v}\right),\\
y_{vv} &= \mathfrak{A}''y_u - 2\frac{\delta_v}{\delta}\cdot y_v - \mathfrak{A}''\frac{\partial}{\partial u}\log\frac{\mathfrak{A}''}{\delta^2}\cdot y,
\end{aligned}
$$

* "Recherches sur le calcul intégral aux différences partielles." Oeuvres de Laplace, t. IX, pp. 29, et seq.

† Koenigs, "Sur les réseaux plans à invariants égaux et les lignes asymptotiques." *Comptes Rendus*, vol. 114 (1892), p. 55.

which is characterized by the relations $a' = b' = 0$, and where δ is a non-vanishing function of u and v.* The integrability conditions of (Y) are obtained from the identical relations

$$(y_{uu})_v = (y_{uv})_u; \quad (y_{uv})_v = (y_{vv})_u.$$

They are given, for the general case, as equations (5) of Wilczynski's paper. Each of the other nets of the Laplace suite has the six corresponding integrability conditions, which are satisfied as a result of those of the original net. We shall, however, use the integrability conditions of the other nets of the suite as being, by their form, better adapted to the simplification of our computations, and shall refer to them as the first, second, etc., the order being understood to be the same as in the case of those of the original net, as given by Wilczynski.

2. *The General Laplace Transform.*—Let us assume that the $(k+1)$th (k a positive integer) transform is non-degenerate,† and has the following coefficients:

$$(4)\qquad \begin{aligned}
a_{k+1} &= \frac{\partial}{\partial u}\log\frac{\mathfrak{A}_k''\,|\,H_k|}{\delta^2}, & b_{k+1} &= \frac{H_k}{\mathfrak{A}_k''},\\
a_{k+1}' &= \frac{\partial}{\partial v}\log|H_k|, & b_{k+1}' &= 0,\\
a_{k+1}'' &= \mathfrak{A}_{k+1}'', & b_{k+1}'' &= \frac{\partial}{\partial v}\log\frac{\mathfrak{A}_k''\,|}{\delta^2},\\
c_{k+1} &= -\frac{H_k}{\mathfrak{A}_k''}\cdot\frac{\partial}{\partial v}\log\frac{\mathfrak{A}_{k-1}''\,|}{H_{k-1}|\,\delta^2},\\
c_{k+1}' &= H_k,\\
c_{k+1}'' &= c_k'' + \frac{\partial^2}{\partial v^2}\log\frac{\mathfrak{A}_{k-1}''\,|}{(H_{k-1}|)^2\delta^2}\\
&\qquad - \frac{\partial}{\partial v}\log\mathfrak{A}_k''\cdot\frac{\partial}{\partial v}\log\frac{\mathfrak{A}_{k-1}''\,|}{H_{k-1}|\,\delta^2},
\end{aligned}$$

* The transformation which accomplishes this, with the help of the integrability conditions, is of the form of the first of (2), and does not alter the net.

† It is easily shown that for the $(k+1)$th transform to degenerate into a single curve it is necessary and sufficient that $\mathfrak{A}_k''H_k = 0$ (cf. (5)), assuming that the kth transform is not degenerate. The assumption that the $(k+1)$th transform is non-degenerate carries with it, of course, the assumption of the non-degeneracy of all the transforms up to the $(k+1)$th.

where

$$H_j = H_{j-1} - \frac{\partial^2}{\partial u \partial v} \log H_{j-1}!,$$

$$\mathfrak{A}_j'' = \frac{\mathfrak{A}_{j-1}''}{H_{j-1}} \left(H_{j-1} - \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{A}_{j-1}''!}{\delta^2} \right), \tag{5}$$

and

$$\theta_j! = \theta_0 \theta_1 \theta_2 \cdots \theta_j, \quad \theta_0 = \theta,$$

(j a positive integer, $\theta = \mathfrak{A}''$ or H).

The coefficients of the first and second transforms, if these transforms are not degenerate, can readily be shown to be of the form (4).

The $(k+2)$th transform is generated by the point $P_{y^{(k+2)}}$, where

$$y^{(k+2)} = y_v^{(k+1)} - \frac{\partial}{\partial v} \log H_k! \cdot y^{(k+1)}. \tag{6}$$

By differentiation of (6) and application of the equations given by (4), we find the relations

$$\begin{aligned} y_u^{(k+2)} &= H_{k+1} y^{(k+1)}, \\ y_v^{(k+2)} &= \mathfrak{A}_{k+1}'' y_u^{(k+1)} + \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\, \delta^2} \cdot y_v^{(k+1)} \\ &\qquad + \left(c_{k+1}'' - \frac{\partial^2}{\partial v^2} \log H_k! \right) y^{(k+1)}. \end{aligned} \tag{7}$$

Except when $\mathfrak{A}_{k+1}'' H_{k+1} = 0$, that is, except when the $(k+2)$th transform is degenerate, we may solve equations (6) and (7) for $y^{(k+1)}$, $y_u^{(k+1)}$, $y_v^{(k+1)}$, obtaining the expressions

$$\begin{aligned} y^{(k+1)} &= \frac{1}{H_{k+1}} y_u^{(k+2)}, \\ y_v^{(k+1)} &= y^{(k+2)} + \frac{1}{H_{k+1}} \frac{\partial}{\partial v} \log H_k! \cdot y_u^{(k+2)}, \\ y_u^{(k+1)} &= \frac{1}{H_{k+1}} \frac{\partial}{\partial u} \log \frac{\mathfrak{A}_{k+1}''!\, H_k!}{\delta^2} \cdot y_u^{(k+2)} \\ &\qquad + \frac{1}{\mathfrak{A}_{k+1}''} y_v^{(k+2)} - \frac{1}{\mathfrak{A}_{k+1}''} \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\, \delta^2} \cdot y^{(k+2)}, \end{aligned} \tag{8}$$

where the fourth integrability condition of $(Y^{(k+1)})$ has been used to obtain the given form of the coefficient of $y_u^{(k+2)}$ in the third equation of (8).

If we differentiate equations (7) again, and make use of (4), we find the following equations:

$$
\begin{aligned}
y_{uu}^{(k+2)} &= (H_{k+1})_u \cdot y^{(k+1)} + H_{k+1} \cdot y_u^{(k+1)}, \\
y_{uv}^{(k+2)} &= (H_{k+1})_v \cdot y^{(k+1)} + H_{k+1} \cdot y_v^{(k+1)}, \\
y_{vv}^{(k+2)} &= \left(c''_{k+1} + \frac{\partial^2}{\partial v^2} \log \frac{\mathfrak{A}_k''!}{(H_k!)^2 \delta^2} \right. \\
&\quad \left. + \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{\delta^2} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\,\delta^2} \right) y_v^{(k+1)} \\
&\quad + \mathfrak{A}''_{k+1} \frac{\partial}{\partial v} \log \frac{\mathfrak{A}''_{k+1}!}{\delta^2} \cdot y_u^{(k+1)} \\
&\quad + \left(c''_{k+1} \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!\, c''_{k+1}}{H_k!\,\delta^2} - \frac{\partial^2}{\partial v^2} \log H_k! \right. \\
&\quad \left. + H_k \mathfrak{A}''_{k+1} \right) y^{(k+1)}.
\end{aligned} \tag{9}
$$

Substitution of (8) in (9) yields the characteristic system of equations of the $(k+2)$th transform, for which the following are the coefficients:

$$
\begin{aligned}
a_{k+2} &= \frac{\partial}{\partial u} \log \frac{\mathfrak{A}''_{k+1}!\, H_{k+1}!}{\delta^2}, & b_{k+2} &= \frac{H_{k+1}}{\mathfrak{A}''_{k+1}}, \\
a'_{k+2} &= \frac{\partial}{\partial v} \log H_{k+1}!, & b'_{k+2} &= 0, \\
a''_{k+2} &= \mathfrak{A}''_{k+2}, & b''_{k+2} &= \frac{\partial}{\partial v} \log \frac{\mathfrak{A}''_{k+1}!}{\delta^2}, \\
c_{k+2} &= -\frac{H_{k+1}}{\mathfrak{A}''_{k+1}} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\,\delta^2}, \\
c'_{k+2} &= H_{k+1}, \\
c''_{k+2} &= c''_{k+1} + \frac{\partial^2}{\partial v^2} \log \frac{\mathfrak{A}_k''!}{(H_k!)^2 \delta^2} \\
&\qquad - \frac{\partial}{\partial v} \log \mathfrak{A}''_{k+1} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\,\delta^2},
\end{aligned} \tag{10}
$$

where the expression for a''_{k+2} (cf. (5)) is given by the fifth integrability condition of $(Y^{(k+1)})$. Comparison of (10) with

(4) shows that we have completed the induction proof that (4) gives the characteristic system $(Y^{(k+1)})$ for the $(k+1)$th transform (k positive).

We may compute the general minus $(k+1)$th transform in the same manner, or may more easily obtain the result by applying the substitution

$$\begin{pmatrix} \mathfrak{A}_j'' & u & c_k'' \\ \mathfrak{B}_{-j} & v & c_k \end{pmatrix} \quad (j = k-1, k, k+1)$$

to the equations given by (4). The coefficients of the minus $(k+1)$th (k positive) turn out to be

$$a_{-(k+1)} = \frac{\partial}{\partial u} \log \frac{\mathfrak{B}_{-k}!}{\delta^2}, \quad b_{-(k+1)} = \mathfrak{B}_{-(k+1)},$$

$$c_{-(k+1)} = c_{-k} + \frac{\partial^2}{\partial u^2} \log \frac{\mathfrak{B}_{-(k-1)}!}{(H_{k-1}!)^2 \delta^2} - \frac{\partial}{\partial u} \log \mathfrak{B}_{-k} \cdot \frac{\partial}{\partial u} \log \frac{\mathfrak{B}_{-(k-1)}!}{H_{k-1}!\, \delta^2},$$

(11)
$$a'_{-(k+1)} = 0, \qquad b'_{-(k+1)} = \frac{\partial}{\partial u} \log H_k!,$$

$$c'_{-(k+1)} = H_k,$$

$$a''_{-(k+1)} = \frac{H_k}{\mathfrak{B}_{-k}}, \quad b''_{-(k+1)} = \frac{\partial}{\partial v} \log \frac{\mathfrak{B}_{-k}!\, H_k!}{\delta^2},$$

$$c''_{-(k+1)} = -\frac{H_k \partial}{\mathfrak{B}_{-k} \partial u} \log \frac{\mathfrak{B}_{-(k-1)}!}{H_{k-1}!\, \delta^2},$$

where

(12)
$$\mathfrak{B}_{-j} = \frac{\mathfrak{B}_{-(j-1)}}{H_{j-1}} \left(H_{j-1} - \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{B}_{-(j-1)}!}{\delta^2} \right)$$

(j any positive integer),

and

$$\mathfrak{B}_{-j}! = \mathfrak{B}_0 \mathfrak{B}_{-1} \mathfrak{B}_{-2} \cdots \mathfrak{B}_{-j}.$$

We have already mentioned the fact that for the $(k+2)$th transform to degenerate, while the $(k+1)$th transform is non-degenerate, it is necessary and sufficient that $\mathfrak{A}_{k+1} = 0$, or $H_{k+1} = 0$. Reference to the third equation of the system determined by (4) shows that *the first of these conditions is*

necessary and sufficient for the curves u = const. of the $(k+1)$th transform to be straight lines. The first equation of (4) shows that *the curves v = const. of the $(k+1)$th transform cannot be straight lines,* since the condition for this, $b_{k+1} = 0$, would make the $(k+1)$th transform degenerate. Similarly, *degeneracy of the minus $(k+2)$th transform is equivalent to $\mathfrak{B}_{-(k+1)} = 0$, or $H_{k+1} = 0$. The first of these conditions is necessary and sufficient for the curves v = const. of the minus $(k+1)$th transform to be straight lines. The curves u = const. of this transform cannot be straight lines.*

3. *Summary of Fundamental Invariants.*—In the paper referred to, Wilczynski has proved the theorem: *If the invariants* $\mathfrak{B} = b$, $\mathfrak{C} = c + a'b + ab' - b'^2 - b_u'$, $\mathfrak{A}' = \frac{2}{3}a' - \frac{1}{3}b'' + \frac{1}{6}(a_v''/a'')$, $\mathfrak{B}' = \frac{2}{3}b' - \frac{1}{3}a + \frac{1}{6}(b_u/b)$, $\mathfrak{C}' = c' + a'b' - \frac{1}{3}(a_v + b_v')$, $\mathfrak{A}'' = a''$, $\mathfrak{C}'' = c'' + a''b' + a'b'' - a'^2 - a_v'$, *of a net are given as functions of u and v, subject to the integrability conditions, the net is determined, except for a projective transformation.** Omitting the details of computation, these fundamental invariants, together with the invariants H and K, for the $(k+1)$th transform have the following expressions:

$$
\begin{aligned}
\mathfrak{B}_{k+1} &= \frac{H_k}{\mathfrak{A}_k''}, \\
\mathfrak{C}_{k+1} &= -\frac{H_k}{\mathfrak{A}_k''} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_{k-1}''!}{(H_{k-1}!)^2 H_k \delta^2}, \\
\mathfrak{A}_{k+1}' &= \frac{1}{6} \frac{\partial}{\partial v} \log \frac{(H_k!)^4 \delta^4 \mathfrak{A}_{k+1}''}{(\mathfrak{A}_k''!)^2}, \\
\mathfrak{B}_{k+1}' &= \frac{1}{6} \frac{\partial}{\partial u} \log \frac{H_k \delta^4}{(\mathfrak{A}_k''!)^2 \mathfrak{A}_k'' (H_k!)^2}, \qquad (13) \\
\mathfrak{C}_{k+1}' &= H_k - \frac{1}{3} \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{A}_k''! \, H_k!}{\delta^2}, \\
\mathfrak{A}_{k+1}'' &= \mathfrak{A}_{k+1}'', \\
\mathfrak{C}_{k+1}'' &= -\mathfrak{A}_{k+1}'' \frac{\partial}{\partial u} \log \frac{\mathfrak{A}_{k+1}''! \, H_k!}{\delta^2}, \\
H_{k+1} &= H_{k+1}, \qquad K_{k+1} = H_k.
\end{aligned}
$$

* P. 485 of Wilczynski's paper.

The corresponding invariants of the minus $(k+1)$th transform are

$$
\begin{aligned}
\mathfrak{B}_{-(k+1)} &= \mathfrak{B}_{-(k+1)}, \\
\mathfrak{C}_{-(k+1)} &= -\mathfrak{B}_{-(k+1)} \frac{\partial}{\partial v} \log \frac{\mathfrak{B}_{-(k+1)}!\, H_k!}{\delta^2}, \\
\mathfrak{A}'_{-(k+1)} &= \frac{1}{6} \frac{\partial}{\partial v} \log \frac{H_k \delta^4}{(\mathfrak{B}_{-k}!)^2 \mathfrak{B}_{-k} (H_k!)^2}, \\
\mathfrak{B}'_{-(k+1)} &= \frac{1}{6} \frac{\partial}{\partial u} \log \frac{(H_k!)^4 \mathfrak{B}_{-(k+1)} \delta^4}{(\mathfrak{B}_{-k}!)^2}, \\
\mathfrak{C}'_{-(k+1)} &= H_k - \frac{1}{3} \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{B}_{-k}!\, H_k!}{\delta^2}, \\
\mathfrak{A}''_{-(k+1)} &= \frac{H_k}{\mathfrak{B}_{-k}}, \\
\mathfrak{C}''_{-(k+1)} &= -\frac{H_k}{\mathfrak{B}_{-k}} \cdot \frac{\partial}{\partial u} \log \frac{\mathfrak{B}_{-(k-1)}!\, H_k}{(H_k!)^2 \delta^2}, \\
H_{-(k+1)} &= H_k, \qquad K_{-(k+1)} = H_{k+1}.
\end{aligned}
\tag{14}
$$

4. *Quasi-periodic Nets.*—In case the ith transform is projectively equivalent to the $(i+j)$th transform (j a positive integer, i a positive or negative integer), the net is said to be quasi-periodic, of period j. Let us assume that $\mathfrak{A}_k''$, $\mathfrak{B}_{-k}$ and H_k are different from zero, so that the $(k+1)$th and minus $(k+1)$th transforms are non-degenerate. A necessary and sufficient condition for a quasi-periodic net, of period $2(k+1)$, is that the fundamental invariants of $(Y^{(k+1)})$ be equal to the corresponding invariants of $(Y^{-(k+1)})$. If such a period exists, however, the invariants H of the two transforms must also be equal. By reference to (5), we see that this condition is

$$
\frac{\partial^2}{\partial u \partial v} \log H_k! = 0, \tag{15}
$$

so that by a transformation of the form $\bar{u} = U(u)$, $\bar{v} = V(v)$ we may make $H_k! = 1$. But (15) also makes the $(k+1)$th and minus $(k+1)$th transforms asymptotic, so that we may take advantage of the following theorem: *If the coefficients a, b, a'', b'', of the form* (3) *of the differential equations of an*

*asymptotic plane net are given as functions of u and v, subject to the integrability conditions, they determine the net except for a projective transformation.** For the case of quasi-period $2(k+1)$, the transformation which makes $H_k! = 1$ also puts the differential equations of the $(k+1)$th and the minus $(k+1)$th transforms, namely, those given by (10) and (11), in the form (3). Hence we obtain the further conditions desired by equating the coefficients a_{k+1}, b_{k+1}, a''_{k+1}, b''_{k+1}, to the corresponding coefficients of the minus $(k+1)$th transform. Remembering that $H_k! = 1$, these conditions are the following:

$$
\begin{aligned}
&\frac{\partial}{\partial u}\log\frac{\mathfrak{A}_k''!}{\delta^2} = \frac{\partial}{\partial u}\log\frac{\mathfrak{B}_{-k}!}{\delta^2},\\
&\frac{H_k}{\mathfrak{A}_k''} = \mathfrak{B}_{-(k+1)},\\
&\mathfrak{A}''_{k+1} = \frac{H_k}{\mathfrak{B}_{-k}},\\
&\frac{\partial}{\partial v}\log\frac{\mathfrak{A}_k''!}{\delta^2} = \frac{\partial}{\partial v}\log\frac{\mathfrak{B}_{-k}!}{\delta^2}.
\end{aligned}
\tag{16}
$$

The first and fourth of these equations imply $\mathfrak{A}_k''!/\mathfrak{B}_{-k}! =$ constant, and we may make this constant equal to unity by a suitable transformation of the independent variables, without violating the condition $H_k! = 1$. The second and third equations of (16) are now equivalent, in view of (5) and (12), so that we have the theorem: *If an asymptotic plane net, for which the $(k+1)$th and minus $(k+1)$th transforms are not degenerate, is quasi-periodic, of period $2(k+1)$, its differential equations may be so written that*

$$
H_k! = 1, \quad \mathfrak{A}_k''! = \mathfrak{B}_{-k}!, \quad \mathfrak{A}''_{k+1} = H_k/\mathfrak{B}_{-k}. \tag{17}
$$

Conversely, any asymptotic plane net, whose $(k+1)$th and minus $(k+1)$th transforms are non-degenerate, and for which equations (17) *hold, is quasi-periodic, of period $2(k+1)$.*

In order to discuss the case of odd quasi-periods, we equate the corresponding fundamental invariants of $(Y^{(-k)})$ and $(Y^{(k+1)})$, to obtain the conditions for a quasi-period $2k+1$. The conditions are the following:

* The truth of this theorem is easily seen by reference to (3).

$$\mathfrak{B}_{-k} = \frac{H_k}{\mathfrak{A}_k''},$$

$$-\mathfrak{B}_{-k}\frac{\partial}{\partial v}\log\frac{\mathfrak{B}_{-k}!\,H_{k-1}!}{\delta^2} = -\frac{H_k}{\mathfrak{A}_k''}\cdot\frac{\partial}{\partial v}\log\frac{\mathfrak{A}_{k-1}''!\,H_k}{(H_k!)^2\delta^2},$$

$$\frac{1}{6}\frac{\partial}{\partial v}\log\frac{\delta^4 H_{k-1}}{(\mathfrak{B}_{-(k-1)}!)^2\mathfrak{B}_{-(k-1)}(H_{k-1}!)^2} = \frac{1}{6}\frac{\partial}{\partial v}\log\frac{(H_k!)^4\delta^4\mathfrak{A}_{k+1}''}{(\mathfrak{A}_k''!)^2},$$

$$\frac{1}{6}\frac{\partial}{\partial u}\log\frac{(H_{k-1}!)^4\delta^4\mathfrak{B}_{-k}}{(\mathfrak{B}_{-(k-1)}!)^2} = \frac{1}{6}\frac{\partial}{\partial u}\log\frac{\delta^4 H_k}{(\mathfrak{A}_k''!)^2\mathfrak{A}_k''(H_k!)^2}, \tag{18}$$

$$H_{k-1} - \frac{1}{3}\frac{\partial^2}{\partial u\partial v}\log\frac{\mathfrak{B}_{-(k-1)}!H_{k-1}!}{\delta^2} = H_k - \frac{1}{3}\frac{\partial^2}{\partial u\partial v}\log\frac{\mathfrak{A}_k''!\,H_k!}{\delta^2},$$

$$\frac{H_{k-1}}{\mathfrak{B}_{-(k-1)}} = \mathfrak{A}_{k+1}'',$$

$$-\frac{H_{k-1}}{\mathfrak{B}_{-(k-1)}}\cdot\frac{\partial}{\partial u}\log\frac{\mathfrak{B}_{-(k-2)}!\,H_{k-1}}{(H_{k-1}!)^2\delta^2} = -\mathfrak{A}_{k+1}''\frac{\partial}{\partial u}\log\frac{\mathfrak{A}_{k+1}''!\,H_k!}{\delta^2}.$$

Since also $H_{-k} = H_{k+1}$, we have, in view of (5),

$$\frac{\partial^2}{\partial u\partial v}\log H_{k-1}!\,H_k! = 0, \tag{19}$$

so that by a suitable transformation of the independent variables we may make

$$H_{k-1}!\,H_k! = 1. \tag{20}$$

By use of (5) and (12), we find that substitution of (20) and the first of (18) in the remaining equations causes (18) to yield only one new condition, namely, $\mathfrak{A}_k''!\,H_{k-1}!/\mathfrak{B}_{-(k-1)}! = \text{constant}$. We may make this constant equal to unity by a transformation of the independent variables, without violating the condition (20). Hence the following theorem results. *If an asymptotic plane net whose* $(k+1)$*th and minus* $(k+1)$*th*

transforms are not degenerate, is quasi-periodic, of period $2k+1$, *its differential equations may be so written that*

$$\begin{gathered} H_{k-1}\,!\,H_k\,! = 1, \quad \mathfrak{A}_k''\mathfrak{B}_{-k} = H_k, \\ \mathfrak{A}_k''\,!\,H_{k-1}\,! = \mathfrak{B}_{-(k-1)}. \end{gathered} \tag{21}$$

Conversely, any asymptotic plane net, whose $(k+1)$*th and minus* $(k+1)$*th transforms are non-degenerate, and for which equations* (21) *hold, is quasi-periodic, of period* $2k+1$.

ANN ARBOR, MICH.,
April, 1916.

CONCERNING HILL'S DERIVATION OF THE LAGRANGE EQUATIONS OF MOTION.

BY PROFESSOR K. P. WILLIAMS.

(Read before the American Mathematical Society, April 22, 1916.)

THERE are two methods of deriving the Lagrange equations of motion that are commonly given in treatises on dynamics. One of the methods makes use of what is known as Hamilton's principle, while the other proceeds directly from D'Alembert's equation by means of a transformation of variables. While the first method leaves little to be desired as regards elegance, it makes use of a principle not essential to an understanding of the equations of motion or of their application. The second method, as usually given, involves a considerable amount of calculation.

In a paper entitled "On the differential equations of dynamics," in the first volume of the *Analyst*,* Hill sought to derive the Lagrange equations from D'Alembert's equation without making use of the details of the calculation above mentioned. For some reason his ideas do not seem to have found their way into the literature of the subject. In the form in which he presented it, Hill's derivation seems to me to be open to criticism on account of some of the assumptions that he makes. It is possible, however, to avoid making these assumptions, and when this is done a very simple and direct derivation of the Lagrange equations is obtained.

We start with D'Alembert's equation

* Collected Papers, vol. 1, pp. 192–194.

$$\sum_{i=1}^{n}\left[\left(m_i\frac{d^2x_i}{dt^2}-X_i\right)\delta x_i+\left(m_i\frac{d^2y_i}{dt^2}-Y_i\right)\delta y_i + \left(m_i\frac{d^2z_i}{dt^2}-Z_i\right)\delta z_i\right]=0,$$

where δx_i, δy_i, δz_i is any virtual displacement. Since

$$\frac{d}{dt}\delta u=\delta\frac{du}{dt} \qquad (u=x_i, y_i, z_i),$$

we obtain from this in the ordinary way

$$(1)\quad \frac{d}{dt}\sum_{i=1}^{n}m_i\left\{\frac{dx_i}{dt}\delta x_i+\frac{dy_i}{dt}\delta y_i+\frac{dz_i}{dt}\delta z_i\right\}-\delta(T-U)=0,$$

where T is the kinetic energy and $-U$ the force function. Let now $q_1, q_2, \cdots, q_k$ be the generalized coordinates of the system. Replacing x_i, y_i, z_i by their values in terms of the q's, the last equation takes the form

$$\frac{d}{dt}\sum_{i=1}^{k}p_i\delta q_i-\delta(\overline{T}-\overline{U})=0,$$

where $p_1, p_2, \cdots, p_k$ are quantities to be determined, and $\overline{T}$ and $\overline{U}$ denote what T and U, respectively, become after the substitution. At this point Hill says: "We can find the value of p_i without actually making the substitution from this consideration; since the original equation contains only the variations δx, δy, δz, etc., without the variations $\delta(dx/dt)$, $\delta(dy/dt)$, $\delta(dz/dt)$, it follows that, in the transformed state, it should contain only the variations δq_i without the variations $\delta(dq_i/dt)$." It scarcely seems legitimate to draw such a conclusion without further examination, especially since Hill expressly gives to the symbol $\delta(dx/dt)$, etc., no quantitative significance.* The hypothesis that must be made in order to evaluate p_i is obtained in the following way.

* The way in which Hill uses the symbol seems to me a little vague. Relative to the relation

$$\frac{d\delta x}{dt}=\delta\frac{dx}{dt}=\delta x'$$

he states: "The reader will see in this only a notational assumption, without quantitative significance, serving merely as machinery of demonstration." He expressly avoids using δx, δy, δz with the significance attached to them in the calculus of variations. This is legitimate in the case of D'Alembert's equation, where they may represent any virtual displacement, but they must be regarded as certain functions of t in equation (1).

The last equation when developed becomes

$$\sum_{i=1}^{k}\left(\frac{dp_i}{dt}\delta q_i + p_i\frac{d}{dt}\delta q_i\right) - \sum_{i=1}^{k}\left(\frac{\partial(\bar{T}-\bar{U})}{\partial q_i}\delta q_i + \frac{\partial \bar{T}}{\partial q_i'}\delta q_i'\right) = 0$$

or

$$(2)\quad \sum_{i=1}^{k}\left[\frac{dp_i}{dt} - \frac{\partial(\bar{T}-\bar{U})}{\partial q_i}\right]\delta q_i + \sum_{i=1}^{k}\left[p_i - \frac{\partial \bar{T}}{\partial q_i'}\right]\frac{d}{dt}\delta q_i = 0,$$

since

$$\delta q_i' = \frac{d}{dt}\delta q_i.$$

Suppose now that the system is holonomic. We can then give to the variations δq_i any values. Let us make each one of them constant throughout the motion, but not zero. The last summation in the equation above then vanishes, so that we have

$$\sum_{i=1}^{k}\left[\frac{dp_i}{dt} - \frac{\partial(\bar{T}-\bar{U})}{\partial q_i}\right]\delta q_i = 0.$$

Since $\delta q_1, \cdots, \delta q_k$ are all independent we then have at once

$$(3)\qquad \frac{dp_i}{dt} - \frac{\partial(\bar{T}-\bar{U})}{\partial q_i} = 0 \quad (i = 1, 2, \cdots, k),$$

an equation entirely independent of the variations δq_i. Equation (2) then becomes

$$\sum_{i=1}^{k}\left[p_i - \frac{\partial \bar{T}}{\partial q_i'}\right]\frac{d}{dt}\delta q_i = 0.$$

Making all the quantities δq_i constant except in turn δq_1, $\delta q_2, \cdots, \delta q_k$, we have finally

$$p_i = \frac{\partial \bar{T}}{\partial q_i'}.$$

Upon substituting in (3) we have the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial q_i'}\right) - \frac{\partial(\bar{T}-\bar{U})}{\partial q_i} = 0 \quad (i = 1, 2, \cdots, k).$$

INDIANA UNIVERSITY.

A SIMPLIFICATION OF THE WHITEHEAD-HUNTINGTON SET OF POSTULATES FOR BOOLEAN ALGEBRAS.

BY DR. B. A. BERNSTEIN.

(Read before the San Francisco Section of the American Mathematical Society, November 20, 1915.)

OF the various sets of postulates that have been given for Boolean logic the most elegant and natural is the set of Huntington's based on Whitehead's "formal laws."* This set may be simplified by reducing the number of its postulates without injuring, the writer feels, the elegance or the naturalness of the original. This reduction is effected by substituting for Huntington's Postulates II_a, II_b, and V the following single postulate:

POSTULATE X. *For any element b in the class there exists an element $\bar{b}$ such that, whatever a is, $a \oplus (b \odot \bar{b}) = a$ and $a \odot (b \oplus \bar{b}) = a$.*

Evidently, Huntington's Postulates II_a, II_b, and V follow from Postulate X, with the help of I_a and I_b.

Evidently, also, Postulate X can be derived from II_a, II_b, and V, with the help of I_a, I_b, III_a, and III_b.

It is of course seen that by adopting Postulate X in place of II_a, II_b, and V, not only is the number of Huntington's postulates reduced from ten to eight, but also the number of postulated special elements is reduced from three ("zero," the "whole," and the "negative") to one (the "negative").

In establishing the independence of the modified set of postulates Huntington's systems for I_a, I_b, IV_a, IV_b, VI can serve for the same numbered postulates in the new set. For Postulate X we can take Huntington's system for V. For III_a and III_b, however, a class of more than two elements is, in each case, necessary. Proof-systems for these two postulates are, respectively, the following:

$\overline{III}_a$.

$\oplus$	e_1	e_2	e_3
e_1	e_1	e_1	e_1
e_2	e_1	e_2	e_2
e_3	e_1	e_3	e_3

$\odot$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_2	e_2
e_3	e_3	e_2	e_3

* See E. V. Huntington, "Sets of independent postulates for the algebra of logic," *Transactions Amer. Math. Society*, vol. 5 (1904), pp. 288–309. The set referred to is the first of the three sets treated by Huntington in his paper.

Here $e_2 \oplus e_3 \neq e_3 \oplus e_2$.

$\overline{\text{III}}_b$.

$\oplus$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_2	e_2
e_3	e_3	e_2	e_3

$\odot$	e_1	e_2	e_3
e_1	e_1	e_1	e_1
e_2	e_1	e_2	e_2
e_3	e_1	e_3	e_3

Here $e_2 \odot e_3 \neq e_3 \odot e_2$.

UNIVERSITY OF CALIFORNIA,
March, 1916.

NOTE ON REGULAR TRANSFORMATIONS.

BY DR. L. L. SILVERMAN.

LET $u(x)$ be bounded and integrable, $0 \leqq x$, and $k(x, y)$ integrable in y for each x, $0 < y \leqq x$; then the transformation*

$$v(x) = \alpha u(x) + \int_0^x k(x, s)u(s)ds \tag{1}$$

is regular if

$$\lim_{x=\infty} u(x)$$

implies the existence of

$$\lim_{x=\infty} v(x)$$

and the equality of the limits. The transformation (1), which depends on the number α and on the function $k(x, y)$, will be denoted by the symbol $[\alpha; k(x, y)]$. Examples of regular transformations are given by $[1; 0]$, which is the identical transformation, and $[0; 1/x]$, which corresponds to the first Hölder mean. In a forthcoming paper† the author discusses conditions on α and $k(x, y)$ for the regularity of the transformation‡ (1), and proves the following theorem:‖

THEOREM 1. *A sufficient condition that $k(x, y)$ defined, $0 < y \leqq x$, and integrable in y for each x, correspond to a*

* It is assumed that the improper integral converges; the lower limit of integration is taken zero for convenience.

† *Transactions*, vol. 17 (1916).

‡ The function $k(x, y)$ in (1) is $(1 - \alpha)$ times the function $k(x, y)$ in the article referred to.

‖ See Theorem III in the article referred to; the numbers a and b of that theorem are here replaced by 0 and a respectively. The right-hand member of the last condition is $1 - \alpha$ instead of unity; see preceding footnote.

regular transformation is

$$\int_0^\infty |k(x, y)|dy \text{ converges}, \quad \lim_{x=\infty} \int_0^\infty |k(x, y)|dy = 0,$$

$$\int_0^\infty |k(x, y)|dy < A, \quad x > 0, \quad \lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where a and A are positive constants.

We wish in this note to consider some special cases of Theorem 1.

THEOREM 2. *Let $k(x, y)$ be defined, $0 < y \leqq x$, and integrable in y for each x; then a sufficient condition* that the transformation $[\alpha; k(x, y)]$ be regular is that*

$$|k(x, y)| \leqq \frac{Mf'(y)}{f(x)}, \quad 0 < y \leqq x, \quad \lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where $f(x)$ is a function continuous, $x \geqq 0$, and having a continuous derivative, $x > 0$, and satisfying the conditions

$$f(x) \geqq 0, \quad x \geqq 0; \quad f'(x) \geqq 0, \quad x > 0; \quad \lim_{x=\infty} f(x) = \infty;$$

and where

$$M \geqq 0.$$

The fourth condition of Theorem 1 is satisfied by hypothesis. The first condition is satisfied since from the hypothesis it follows that the integral of $k(x, y)$ converges absolutely. Furthermore,

$$\int_0^\infty |k(x, y)|dy \leqq \frac{Mf(a)}{f(x)}, \quad x > 0.$$

The second and third conditions of Theorem 1 follow at once from this inequality. Thus all the conditions of Theorem 1 are satisfied.

COROLLARY 1. *A sufficient condition that $k(x, y)$ defined, $0 < y \leqq x$, and integrable in y for each x, correspond to a regular transformation is*

$$|k(x, y)| \leqq \frac{M}{(1 + y) \log (1 + x)}, \quad 0 < y \leqq x,$$

$$\lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where $M \geqq 0$.

* The convergence of the integral in the second condition—in fact, its absolute convergence—follows from the first condition.

Taking $f(x) = \log(1 + x)$, it is seen that the conditions of Theorem 2 are satisfied.

COROLLARY 2. *A sufficient condition that $k(x, y)$ defined, $0 < y \leqq x$, and integrable in y for each x, correspond to a regular transformation is*

$$|k(x, y)| \leqq \frac{M}{x^{1-p}y^p}, \quad 0 < y \leqq x, \quad \lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where p and M are constants. $0 \leqq p < 1$, $M \geqq 0$.

Taking $f(x) = x^{1-p}$, it is seen that the conditions of Theorem 2 are satisfied.

Similar theorems hold for transformations of sequences.

CORNELL UNIVERSITY.

SHORTER NOTICES.

Contributions to the Founding of the Theory of Transfinite Numbers. By GEORG CANTOR. Translated, and provided with an introduction and notes, by PHILIP E. B. JOURDAIN. Chicago and London, Open Court Publishing Company, 1915. ix+211 pp.

THIS volume contains a translation of G. Cantor's two fundamental memoirs on transfinite numbers which appeared in the *Mathematische Annalen* for 1895 and 1897 under the title "Beiträge zur Begründung der transfiniten Mengenlehre." The translator has changed the title to that given above because "these memoirs are chiefly occupied with the investigation of the various transfinite cardinal and ordinal numbers." The book is put forth by the publishers as number 1 of "The Open Court Series of Classics of Science and Philosophy."

It is not too much to say that the work of Cantor on the theory of classes of points has brought about both a mathematical and a philosophical revolution, that in philosophy perhaps being even greater than that in mathematics, notwithstanding the fact that "these theories of Cantor are permeating modern mathematics."*

In the opinion of the translator K. Weierstrass, R. Dedekind, and G. Cantor are the three men who have exerted the most marked influence on modern pure mathematics and indirectly on the modern logic and philosophy which abut on it. In

* E. H. Moore, Introduction to a Form of General Analysis, p. 2.

order that the work of Cantor—and, in particular, the two memoirs here translated—may be best understood it is desirable on the one hand to compare it with the work of Dedekind which has developed along a parallel direction and on the other hand to trace its origin backwards through the earlier theory of functions and especially through the work of Weierstrass. To facilitate the latter historical study this book is provided with an excellent introduction (pages 1–82), based in large part on the translator's "Development of the theory of transfinite numbers," published in the *Archiv der Mathematik und Physik* in 1906, 1909, 1910, and 1913.

This introduction begins with a brief account of the contribution of Fourier toward the general definition of function, followed by remarks concerning the labors of Dirichlet, Cauchy, Riemann, and Hankel in the theory of functions as well as by some further discussion of the theory of Fourier series. A fuller account is then given (pages 10–23) of the contributions of Weierstrass to the theory of functions. The remainder of the introduction (pages 23–82) is devoted to an illuminating account of the development of Cantor's ideas as seen from his publications prior to 1895.

The translation of Cantor's articles covers pages 85–201. On the whole the work of the translator is done in a satisfactory manner. One regrets, however, that the term "power" is used to translate both "Potenz" and "Mächtigkeit," thus introducing an ambiguity in the English which is unnecessary and is absent from the German. Sometimes (as on pages 97, 178) the translator finds it necessary to introduce a footnote to remove the ambiguity.

Brief notes at the end (pages 202–208) contain a short but valuable account of the development of the theory of transfinite numbers since 1897.

A few misprints and slips need correction, as follows: on page 16, line 5 below for s_{n+} read $s_{n+\gamma}$; on page 31, lines 8 and 9, for P', P'', P read P'', P''', P'; on page 33, line 8 below, for *consists of* read *contains*; on page 47, end of line 13, insert *at*; on page 69, line 1, for *in* read *is*; at the middle of page 71, for last exponent 3 read 2; on page 96, line 4, after $x \geqq$ insert 0; on page 96, line 9, for $2^{\aleph}$ read $2^{\aleph_0}$; on page 109, line 12 below, for $\aleph$ read $\aleph_0$; on page 184, line 3 below, for the first ω read ω^{ω_0}; on page 194, line 3, for $\alpha^{(\rho_0}$ read $\alpha^{(\rho_0)}$; on page 196, line 11 below, for ω^{γ} read ω^{γ_ν}.

In making this fundamental work of Cantor readily accessible to a wider range of English readers both the translator and the publishers have rendered a useful service in the development of science.

R. D. CARMICHAEL.

Euclid's Book on Divisions of Figures (περὶ διαιρέσεων βιβλίον), with a restoration based on Woepcke's text and on the Practica Geometriæ of Leonardo Pisano. By RAYMOND CLARE ARCHIBALD. Cambridge, Eng., University Press, 1915. ix+88 pp.

OF the nine works attributed to Euclid the "Elements" is, of course, by far the most important and the most widely known. The "Data" is known to us through the τόπος ἀναλυόμενος of Pappus, as stated in the Commandino edition of 1660, page 241; the "Porisms" was restored by Chasles, and earlier by Robert Simson; the "Optics" was known to earlier scholars through Theon, and has recently appeared in a modern edition through the labors of Heiberg; the "Phænomena" is nearly complete and was edited by Menge; the "Conics" is lost, except as part of it may have been embodied in the works of Apollonius; and the "Pseudaria" and "Surface Loci" are known only through fragments. The ninth work, entitled "On Divisions" (of figures), was for a long time known only through references by Proclus, but in 1570 it appeared in print under the editorship of John Dee and Federico (sometimes printed Federigo) Commandino in Latin translation from the Arabic. In 1851 Woepcke found an Arabic manuscript of the work at Paris, and this was published in translation in the *Journal Asiatique*.

It seems that John Dee, when he visited Commandino at Urbino in 1563, gave to the latter a Latin manuscript of the work as translated into Arabic by one Muhammed Bagdedinus, and this together with an Italian version was published seven years later. An English translation appeared in London in 1660 and again in 1661. David Gregory included the Latin text in his edition of Euclid in 1703 with the statement: "Joannes Dee Londinensis, cum Librum de Divisionibus superficierum, Machometo Bagdedino (qui floruisse creditur seculo Christi decimo) vulgo adscriptum, ex Arabico (uti credo, licet hoc expresse non dicat) in Latinum verteret." As to the conjectured date of "Machometo Bagdedino" it may be said in

passing, that Professor Archibald believes, with Suter, that he was the Muhammed of Bagdad who died in 1141, and hence that he was a scholar of the eleventh and twelfth centuries.

Professor Archibald has, with great perseverance and scholarship, cleared up a number of points in connection with this translation. In the first place he has shown that Suter and Steinschneider have not considered their statements concerning it with their usual care. For example, it has been commonly asserted that Dee probably copied a Latin translation in the Cottonian collection, whereas it is here shown that he did not do so, and that no such translation in complete form was ever in the British Museum. It is very doubtful whether a translation made by Gherardo of Cremona was the one referred to in the Cottonian catalogue made by Smith in 1696. At any rate this particular manuscript was not in Gherardo's handwriting since it was of about the fourteenth century. The fact is that Dee owned a manuscript of the work itself, possibly a copy of Gherardo's translations, and very likely he owned another besides the one which (maybe as a duplicate) he gave to Commandino.

Professor Archibald's plan in editing the work was to translate literally everything in Woepcke's French translation from the Arabic manuscript in the Bibliothèque nationale, to reproduce Fibonacci's proofs and constructions as set forth in his "Practica Geometriæ" of 1220, and to show the correspondence of the Muhammed-Commandino treatise with the Euclid text and with Fibonacci. He has also shown the relation of the "Geometria vel De Triangulis" of Jordanus Nemorarius to the work in question, revealing some interesting and significant facts. It would be well worth the attention of some scholar to consider the "De Numeris Datis" in the same spirit.

Of the work itself this is not the place to speak, further than to say that it has to do with such divisions of plane figures as the separation of a given triangle into two equal parts by a line which passes through a point situated in the interior of the triangle, and to call attention to the fact that this is the source of a number of such problems, some of which played a considerable rôle in the older treatises on surveying. But of the work of editing the text it may be said that it is a perfectly appropriate compliment to pay both to Professor Archibald and to Sir

Thomas Heath to say that the care shown by each, in the editing of the classics to which their names attach, is on a par with that shown by the other. Certainly we have not had any such work done before in this country in the editing of any Greek mathematical text, and the thanks and appreciation of Professor Archibald's colleagues will go out to him in abundant measure for his excellent contribution to the literature of the subject.

Not the least of the commendable features of the edition under review is the bibliography of works relating to the division of figures. How complete this is it is difficult to say, since one would have to go through a large amount of material, as has evidently been done in this case, to determine just where to find points of contact with the "De Divisionibus." At any rate the list is a very helpful one and adds materially to the value of the work.

The publishers have allowed the printing of a copious index of names, and have issued the work in the dignified form which always characterizes the output of the Cambridge University Press.

DAVID EUGENE SMITH.

Analytic Geometry. By H. B. PHILLIPS. New York, John Wiley and Sons, 1915. vii+197 pp.

IN the preface of this book the author expresses the belief that for engineering students "a short course in analytic geometry is essential"; and "he has, therefore, written this text to supply a course that will equip the student for work in calculus and engineering without burdening him with a mass of detail useful only to the student of mathematics for its own sake." At first glance the comparatively small number of pages would seem to promise such a short course. But a closer examination led the present writer to the opinion that the apparent brevity was achieved by condensation, and that it would require as much time to cover these 197 pages as to cover say 300 pages of many other texts. Except for the omission of some of the special properties of the conics, it did not seem quite obvious that the student's burden of a mass of detail was conspicuously lightened.

On the other hand, the author at times assumes a clearness of mathematical vision and a facility in technique on the part of the student which would be eminently desirable, but

which we fail to find in a majority of our college students. For instance, after learning (page 94) that "a curve is symmetrical with respect to the origin if all the terms in its equation are of even degree or if all are of odd degree," we find the following example (I quote the text exactly):

"$y = x^3 - 3x^2 + 3x + 1$. This equation can be written

$$y - 2 = (x - 1)^3.$$

The expressions $y - 2$ and $x - 1$ are the coordinates of a point $P(x, y)$ relative to the lines $y = 2$, $x = 1$ used as axes. Since the equation contains only odd powers of $y - 2$ and $x - 1$, the curve is symmetrical with respect to the point (1, 2)." This occurs 50 pages earlier than the treatment of transformations to parallel axes. At such points the ordinary student will certainly need the help of the teacher. However, many teachers will find this no objection to the book, but will prefer to use a text which leaves to the teacher some further function than the assigning of lessons and the conducting of recitations. Teachers with ideas of their own are sometimes hampered by a text too liberal in its explanations.

The first chapter, on algebraic principles, furnishes a good review of those parts of algebra which are most necessary for what follows. One is pleased to find a discussion of inequalities and of homogeneous linear equations. In the second chapter, introducing rectangular coordinates, there are six excellent pages on vectors. The straight line and circle are treated in chapter three, which is short but would seem to furnish the student with the necessary working equipment. A very good feature is the clear exposition of the geometrical significance of the sign of the expression $Ax + By + C$; but it is to be regretted that the idea was not used to the best advantage later; as, for instance, in connection with the distance formula $(Ax_1 + By_1 + C)/ \pm \sqrt{A^2 + B^2}$, and the definition of the hyperbola.

After a somewhat unusual treatment of the conics in chapter four, one finds a very interesting chapter on graphs and empirical equations. In addition to the usual curve drawing, the problem of fitting a curve to a number of given pairs of values of the variables is discussed, with a number of exercises drawn from mechanics, electricity, chemistry, etc. Chapters six, seven, and eight are devoted respectively to polar coordinates, parametric representation, and transformations. The

chapter on parametric representation, with the development of the equations of the cycloidal and other curves, is one of the excellent features of the book. The last three chapters present very briefly the first elements of the analytic geometry of space, treating the plane, straight line, sphere, cylindrical surfaces, surfaces of revolution, and the different quadric surfaces in their simplest positions. The helix is discussed in a very short paragraph on parametric equations of space curves.

The author's treatment of conics should have special mention. The ellipse, parabola, and hyperbola are defined as follows:

"If a circle is deformed in such a way that the distances of its points from a fixed diameter are all changed in the same ratio, the resulting curve is called an ellipse.

"Let LK, RS be perpendicular lines and MP, NP perpendiculars from any point P to them. If a is constant and NP considered positive when P is on one side of RS, negative when on the other, the locus of points P such that

$$MP^2 = a \cdot NP$$

is called a parabola.

"Let KL and RS be two straight lines intersecting in C, PM and PN the perpendiculars from any point P to these lines. Let MP be considered positive when P is on one side of KL, negative when on the other side. Similarly, let NP be positive when P is on one side of RS, negative when on the other side. A hyperbola is the locus of points P such that the product

$$MP \cdot NP = \text{constant.}"$$

The general definition of a conic, and the classification with respect to the eccentricity are given later in the chapter on polar coordinates. There is no question here of correctness or incorrectness, but merely one of taste and expediency. Many teachers would prefer that the conics should be first introduced by a general definition; and many others, willing to use independent definitions of the three types, would prefer other definitions than those which the author has seen fit to use. Immediately following the definition of the parabola we read: "A parabola is thus a locus of points the squares of whose distances from one of two perpendicular lines are proportional to their distances from the other. The complete locus of

such points is two parabolas, one on each side of *RS*." A similar statement follows the definition of the hyperbola. Is it not unfortunate to speak of a locus which is not a complete locus?

The definition of equivalence of sets of equations (page 3) is somewhat vague; and it hardly seems wise to say that the equation $(x + y)(x - 2y) = 0$ is equivalent to the two equations $x + y = 0$ and $x - 2y = 0$, even though the sense in which this is meant is immediately explained.

The statement (page 15) that "a set of homogeneous equations can often be solved for the ratios of the variables when there are not enough equations to determine the exact values" might seem to imply that the "exact" values could be determined if there were enough equations.

In chapter five there is a paragraph on "infinite values" which reminds one of the school algebras of the last generation. It seemed to the present writer to be a really serious defect in what is in many respects an excellent book.

The mechanical features of the book are attractive, the figures (with a few exceptions) are accurate, and the typographical work is free from errors.

WALTER B. CARVER.

A Budget of Paradoxes. By AUGUSTUS DE MORGAN. Reprinted, with the author's additions, from the *Athenæum*. Second edition, edited by DAVID EUGENE SMITH. Two volumes, I, viii+402 pp.; II, 387 pp. Chicago, The Open Court Publishing Co., 1915. Price, $3.50 per volume.

THE first edition of this interesting work by Augustus De Morgan (1806–1871) appeared in 1872, after the author's death, under the editorship of his widow, Sophia De Morgan. Some ten years later Mrs. De Morgan wrote a "Memoir of Augustus De Morgan," which is worthy of mention in connection with the "Budget of Paradoxes." De Morgan's articles which constitute the present work appeared from time to time, in the years from 1863 (Oct. 10) to 1866 (Dec. 1), in the London *Athenæum*. From other facts which we have concerning the life of De Morgan it appears that some of the popular writing which he did, for encyclopedias and for journals, was stimulated by financial pressure; at this distance we can properly rejoice at the conditions which fostered the growth of the present work.

The wide range of interests of De Morgan is nowhere so well shown as in the somewhat random discussions of the "Budget of Paradoxes." In particular his interest in the history of mathematics, and in the history of science, is revealed again and again in these pages. De Morgan holds that a reasonable familiarity with the development of any field of scientific research is an indispensable, necessary condition for a contribution to the field. "All the men who are now called discoverers, in every matter ruled by thought, have been men versed in the minds of their predecessors, and learned in what had been before them. There is not one exception. I do not say that every man has made direct acquaintance with the whole of his mental ancestry; many have, as I may say, only known their grandfathers by the report of their fathers. But even on this point it is remarkable how many of the greatest names in all departments of knowledge have been real antiquaries in their several subjects.

"I may cite, among those who have wrought strongly upon opinion or practise in science, Aristotle, Plato, Ptolemy, Euclid, Archimedes, Roger Bacon, Copernicus, Francis Bacon, Ramus, Tycho Brahé, Galileo, Napier, Descartes, Leibnitz, Newton, Locke. I take none but names known out of their fields of work; and all were learned as well as sagacious. I have chosen my instances: if any one will undertake to show a person of little or no knowledge who has established himself in a great matter of pure thought, let him bring forward his man, and we shall see."

Taking into account the special interests of De Morgan and his great activity in the popularizing of mathematics the selection of David Eugene Smith as editor of this edition was almost inevitable. Between the time of the English mathematician and the present time no one could have been found more admirably fitted by nature and by training to edit the "Budget" than Professor Smith. The similarity of his literary and public activity to De Morgan's is striking; both men have been widely known as the authors of elementary textbooks of unusual excellence, both have acted as editors of the mathematical department of encyclopedias and dictionaries, both have been energetic collectors of old mathematical books and other mathematical material, both have made notable contributions to the history and bibliography of mathematics, and both have been distinguished by a wide and human

interest in mathematics. Smith's "Rara Arithmetica," the illuminating, descriptive catalogue of Mr. G. A. Plimpton's unparalleled collection of arithmetical books and manuscripts, is a continuation and extension of De Morgan's bibliographical work, "Arithmetical Works from the Invention of Printing to the Present Time" (London, 1847).

The intention of De Morgan in publishing the "Budget" was definitely stated to be "to enable those who have been puzzled by one or two discoverers to see how they look in a lump." By "discoverers," here, is meant paradoxical discoverers. For De Morgan the paradox "is something which is apart from general opinion, either in subject-matter, method, or conclusion." That no disparagement is implied necessarily in the designation "paradoxer" is indicated by the fact that De Morgan refers to Copernicus and Galileo as paradoxers, and includes the discovery of the planet Neptune by Le Verrier as a paradox. A distinction is drawn between two types of paradoxer, as follows: "The manner in which a paradoxer will show himself, as to sense or nonsense, will not depend upon what he maintains, but upon whether he has or has not made a sufficient knowledge of what has been done by others, *especially as to the mode of doing it*, a preliminary to inventing knowledge for himself." However it must be stated that the particular interest of De Morgan in this work is in the nonsense type of paradox and paradoxer.

In mathematics the old problems of the squaring of the circle, the duplication of the cube, the trisection of the angle, and numerical juggling with 666, the number of the beast, contribute most largely to the occupation of neophytes who "in a moment by a lucky thought" wish to enter their names upon the limited roll of great mathematicians. Others, and their kind is not yet extinct, have the notion that either at home or abroad there is a great reward offered for the squaring of the circle; some even have deluded themselves into thinking that this financial reward is involved in a university professorship. What could be more illuminating as to the ignorance of these paradoxers! Astronomy contributes its fair quota to the "Budget"; religion, philosophy, and medicine, too, have their pseudo-scientists who without knowing "what has been done by others" wish to revolutionize the established order. Would that all this were, indeed, a closed book. In mathematics trisectors are fairly common, and have persuaded journals,

even with scientific titles, to publish nonsense; on Fermat's theorem several hundred articles have been printed and others are now being written by paradoxers whose tense interest in the subject is occasioned rather by a certain familiarity with the power of the $25,000, an actual prize, than by any familiarity with the powers of integers; what nonsense emanates even from a distinguished seat of learning—absolutely, let us note, without official sanction—concerning a prodigy lecturing upon the fourth dimension; what a priori philosophical nonsense, based upon ignorance of "what has been done by others," has been published concerning the nature of the number idea. A modern De Morgan would, in two volumes like these, have room only for titles of published nonsense.

The present edition of the "Budget" will prove of considerable value to public libraries as a work of reference; the value would be greatly increased by an analytical table of contents, presenting the titles of the articles comprising the work, and giving, possibly, a summary of the articles arranged with reference to subject-matter. Mathematicians and astronomers have a particular interest in the work since so much of the material is germane to their fields. To that wide circle of readers who have enjoyed the artistic ramblings of William De Morgan this work by the famous novelist's father will prove entertaining, for the peculiar literary charm of the son seems to be a direct transmission from the father.

Typographically and otherwise the book is up to the high standard which has been set by the publications of the Open Court Company. The reader who takes the volumes in hand has real pleasure in store: we commend the work to all of those who take a kindly interest in the frailty, as well as in the greatness, of their fellows.

The errors of one sort and another are so difficult to find that it seems desirable to mention those which have caught the eye of the reviewer. In the Preface, I, page iv, the reference in the last line should be to page 280, not 281, of the second volume. The one good "i" is dropped out of an "acquaintance" on page 5, volume I. From Smith's "Rara Arithmetica" three or four errors should be corrected in the title of Robert Recorde's "The Whetstone of Witte" (II, page 328): The whetstone of witte . . . containyng . . . Cossike . . . Nombers. The date of death of Sacrobosco, given as 1256, (I., page 360) is not known. LOUIS C. KARPINSKI.

NOTES.

THE March number (series 2, volume 17, number 3) of the *Annals of Mathematics* contains the following papers: "An isomorphism between theta characteristics and the $(2p+2)$-point," by A. B. COBLE; "On certain real solutions of Babbage's functional equation," by J. F. RITT; "Note on the preceding paper," by A. A. BENNETT; "An elementary exposition of the theory of the gamma function" (authorized translation from the Danish by T. H. GRONWALL), by J. L. W. V. JENSEN.

THE first four numbers of volume two of the *Proceedings of the National Academy of Sciences* contain: "Upper limit of the degree of transitivity of a substitution group," by G. A. MILLER; "An extension of Feuerbach's theorem," by FRANK MORLEY; "Deformations of transformations of Ribaucour," by L. P. EISENHART; "On the linear dependence of functions of several variables, and certain completely integrable systems of partial differential equations," by G. M. GREEN; "Point sets and allied Cremona groups (part II)," by A. B. COBLE; "On a theorem of Lucas," by M. B. PORTER; "Interpretation of the simplest integral invariant of projective geometry," by E. J. WILCZYNSKI.

UNIVERSITY OF CHICAGO.—The following courses are announced for the summer quarter, June 19–September 1: By Professor G. A. BLISS: Theory of functions of a real variable.—By Professor L. E. DICKSON: Substitution groups and algebraic equations, solution of numerical equations (first half); Determinants and symmetric functions (second half).—By Dr. C. R. DINES (of Dartmouth College): Differential equations.—By Professor W. D. MACMILLAN: Introduction to celestial mechanics.—By Professor E. H. MOORE: Integral equations in general analysis (first half); Limits (second half).—By Professor F. R. MOULTON: Theory of functions of infinitely many variables (second half); Series (second half).—By Professor A. RANUM (of Cornell University): Metric differential geometry.—By Professor H. E. SLAUGHT: Elliptic integrals.—By Professor J. W. A. YOUNG: Selected topics in mathematics.

THE following advanced mathematical courses are announced for the academic year 1916–1917:

UNIVERSITY OF CHICAGO.—By Professor G. A. BLISS: Theory of functions of a real variable (a); Calculus of variations (w); Functions of lines (s).—By Professor L. E. DICKSON: Theory of numbers (a); Algebraic numbers (w); Linear algebra (s); Theory of equations (w); Solid analytics (s).—By Professor K. LAVES: Analytic mechanics (a, w).—By Professor A. C. LUNN: Vector analysis (a); Applications to electromagnetics (w); Advanced calculus (a); Geometric introduction to the theory of the complex variable (w).—By Professor W. D. MACMILLAN: Introduction to celestial mechanics (w, s).—By Professor E. H. MOORE: Integral equations in general analysis (a, w, s); Seminar on the foundations of analysis (a, w).—By Professor F. R. MOULTON: Modern theories of analytic differential equations with applications to celestial mechanics and periodic orbits (a, w); Lunar theory (w); Application of the method of periodic orbits to the lunar theory (a).—By Professor H. E. SLAUGHT: Differential equations (a).—By Professor E. J. WILCZYNSKI: Projective geometry (a, w).—By Professor J. W. A. YOUNG: Limits and series (s).

COLUMBIA UNIVERSITY.—By Professor T. S. FISKE: Theory of functions, four hours.—By Professor M. W. HASKELL: Higher plane curves, three hours, first semester; Continuous groups, three hours, first semester.—By Professor F. N. COLE: Algebra, four hours.—By Professor JAMES MACLAY: Theory of numbers, three hours.—By Professor D. E. SMITH: History of mathematics, three hours; Seminar in the teaching of mathematics.—By Professor EDWARD KASNER: Contact transformations, two hours, second semester; Seminar in differential geometry.—By Professor W. B. FITE: Divergent series, three hours, second semester.—By Professor H. E. HAWKES: Differential geometry of curves, three hours, second semester.

JOHNS HOPKINS UNIVERSITY.—By Professor F. MORLEY: Higher geometry (first term), three hours; Theory of functions (second term), three hours.—By Professor A. B. COBLE: Finite groups, two hours; Probabilities (second term), two hours.—By Professor A. COHEN: Theory of numbers, two hours; Theory of functions, two hours.—By Dr. H. BATEMAN: Differential equations of physics, two hours.

UNIVERSITY OF ILLINOIS.—By Professor E. J. TOWNSEND: Functions of real variables, three hours; Differential equations, three hours.—By Professor G. A. MILLER: Elementary theory of groups, three hours; Theory of equations, three hours, first semester.—By Professor H. L. RIETZ: Actuarial theory, three hours.—By Professor J. B. SHAW: Vector methods, three hours.—By Professor C. H. SISAM: Invariants and higher plane curves, three hours; Solid analytic geometry, three hours, second semester.—By Professor A. EMCH: Automorphic functions, three hours.—By Professor R. D. CARMICHAEL: Theory of linear differential equations, three hours.—By Professor A. R. CRATHORNE: Projective geometry, three hours.—By Dr. E. B. LYTLE: History of mathematics, two hours, first semester.—By Dr. A. J. KEMPNER: Modern algebra, three hours.

DR. GEORGE SARTON, of Ghent, editor of *Isis*, has been appointed lecturer on the history of science at Harvard University. In the academic year 1916–1917 he will give two courses, one on "The origin and development of Greek science" and the other on the "Principles of mathematics historically considered." In the year 1917–1918 he will lecture on Hellenic science and on the principles of mechanics historically considered. He will also lecture on the history of science at the Lowell Institute in Boston.

THE Smith prizemen for the year 1916, announced by the University of Cambridge, are as follows: H. M. GARNER, of St. John's College, for his essay, "On orbital oscillation about the equatorial triangular configuration in the problem of three bodies"; G. P. THOMPSON, of Corpus Christi College, for his essay, "On aeroplane problems."

THE Rayleigh prize for 1916 has been awarded to W. M. SMART, of Trinity College, for his essay, "Libration on the Trojan planets."

PROFESSOR C. J. DE LA VALLÉE-POUSSIN, of the University of Louvain, has been elected to membership in the Paris academy of sciences.

PROFESSOR G. A. BLISS, of the University of Chicago, has been elected a member of the National academy of sciences.

PROFESSORS M. BÔCHER, of Harvard University, and F. R. MOULTON, of the University of Chicago, have been elected to membership in the American philosophical society. Professor J. D. v. d. WAALS, of Amsterdam, has been elected a foreign member of the society.

PROFESSOR C. J. KEYSER, of Columbia University, and Professor M. W. HASKELL, of the University of California, will exchange professorships during the first term of 1916–1917.

MR. R. W. DICKEY has been appointed associate professor of mathematics at Washington and Lee University.

DR. J. I. TRACY, of Yale University, has been promoted to an assistant professorship of mathematics.

DR. F. J. MCMACKIN, of Columbia University, has been appointed instructor in mathematics at Dartmouth College.

AT the University of Oklahoma Mr. C. T. LEVY, of the University of California, has been appointed instructor in mathematics. Mr. E. D. MEACHAM, instructor in mathematics, will spend a year's leave of absence in study at Harvard University.

PROFESSOR WEBSTER WELLS, for many years instructor and professor of mathematics in the Massachusetts Institute of Technology and author of several mathematical textbooks, died at Boston on May 23 at the age of sixty-five years. Professor Wells retired from active service in 1911. He was a member of the American Mathematical Society from 1891.

CATALOGUES of books:—Carnegie Institution of Washington, List of publications to March 1, 1916.—W. Heffer and Sons, Cambridge, England, catalogue 148, including selections from the library of the late E. J. Routh.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BECKER (O.). Ueber die Zerlegung eines Polygons in exklusive Dreiecke auf Grund der ebenen Axiome der Verknüpfung und Anordnung. Leipzig, 1914. 8vo. 71 pp. M. 2.00

BIEBERBACH (L.). Einführung in die konforme Abbildung. Leipzig, Göschen, 1915. M. 0.90

——. Ueber die Grundlagen der modernen Mathematik. Leipzig, 1914. 4to. 6 pp.

DARBOUX (G.). Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. Deuxième édition, 2e partie. Paris, Gauthier-Villars, 1915. Royal 8vo. 8+580 pp. Fr. 20.00

DINI (U.). Calcolo integrale. 1a e 2a parte. Pisa, Nistri, 1909–1915. 13+1060 pp.

DUNKEL (O.). See GOURSAT (E.).

FRICKE (R.). Die elliptischen Funktionen und ihre Anwendungen. In 3 Teilen. 1ter Teil: Die funktionentheoretischen und analytischen Grundlagen. Leipzig, Teubner, 1916. 8vo. 10+500 pp. Leinwand. M. 24.00

GOURSAT (E.). A course in mathematical analysis. Functions of a complex variable, being Part 1 of Volume 2. Translated from the second French edition by E. R. Hedrick and O. Dunkel. Boston, Ginn, 1916. 8vo. 10+259 pp. $2.75

HAMBURGER (H.). Ueber die Integration linearer homogener Differentialgleichungen. München, 1914. 4to. 69 pp.

HARST (J. H. VAN DER). Eenige ontaardingen van den harmonischen complex. Leiden, 1915. 8vo. 137 pp.

HEDRICK (E. R.). See GOURSAT (E.).

HOORN (J. VAN). Over de transfinite getallen en de leer van het kontinuum. Leiden, 1915. 4to. 111 pp.

HURWITZ (L.). Ueber den Zusammenhang der Drehungen des Raumes mit dem Kugelkreise. Königsberg, 1914. 8vo. 77 pp.

KILLERMAN (A.). Beweis des Fermatschen Satzes. Nürnberg, 1916. M. 3.00

KLEIN (F.). Bericht über den heutigen Zustand des mathematischen Unterrichts an der Universität Göttingen. Leipzig, Teubner, 1915. M. 0.40

KNOTT (C. G.). See NAPIER.

LÉRY (G.). Sur la fonction de Green pour un contour algébrique. Paris, Gauthier-Villars, 1915. 4to. 88 pp. Fr. 5.00

NAPIER tercentenary memorial volume; being papers contributed to the tercentary congress held at Edinburgh in July 1914. Edited by C. G. Knott. New York, Longmans, 1915. 4to. 12+441 pp. $7.00

PFLÜGER (J.). Die Formschönheit einfacher geometrischer Gebilde. Stuttgart, Metzler, 1915.

PICARD (E.). L'histoire des sciences et les prétensions de la science allemande. (Pour la Vérité.) Paris, Perrin, 1916. 16mo. 48 pp.

PINCHERLE (S.). Algebra complementare. Parte II: Teoria delle equazioni. 3a edizione, riveduta e corretta. (Manuali Hoepli, serie scientifica, No. 145.) Milano, Hoepli, 1916. 24mo. 167 pp. L. 1.50

——. Lezioni di calcolo infinitesimale. Bologna, Zanicelli, 1915. 15+774 pp.

RAPP (P.). Ueber Gruppen vom Rang Null. Tübingen, 1914. 8vo. 35 pp.

ROTTSIEPER (W.). Graphische Lösung einer Randwertaufgabe der Gleichung $\Delta u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$. Göttingen, 1914. 8vo. 45 pp. M. 2.00

SCHMIDT (C. P.). Kulturhistorische Beiträge zur Kenntnis des griechischen und römischen Altertums. 1tes Heft: Zur Entstehung und Terminologie der elementaren Mathematik. 2te verbesserte und stark vermehrte Auflage. Leipzig, Dürrsche Buchhandlung, 1914. 8vo. 16+269 pp.

TORHORST (A.). Die rationale Normalfläche vierter Ordnung des R_5 und ihre Projektionen in den vier- und dreidimensionalen Raum. (Diss.) Bonn, 1915.

VOOREN (W. L. VAN DE). Eene bijdrage tot de kennis der kettingbreuken van Stieltjes. Leiden, 1915. Gr. 8vo. 12+102 pp.

WIEN (W.). Die neuere Entwicklung unserer Universitäten und ihre Stellung im deutschen Geistesleben. Rede für den Festakt in der neuen Universität am 29 Juli 1914 zur Feier der hundertjährigen Zugehörigkeit Würzburgs zu Bayern. Leipzig, 1915. M. 1.00

WILLIGENS (C. L.). Transformation de la fonction modulaire $\eta(\omega)$ ainsi que d'une généralization de cette fonction. Genève, 1914. 8vo. 80 pp.

II. ELEMENTARY MATHEMATICS.

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